QUANTITATIVE UNIQUE CONTINUATION: AN INTRODUCTION AFTER A. LOGUNOV - E. MALINNIKOVA

1. Notations

In what follows, we shall denote by $A(x)$ a real $d \times d$ symmetric matrix defined in a ball $B_{R_0} = \{x \in \mathbf{R}^d : |x| < R_0\}$, with $W^{1,\infty}$ entries, uniformly elliptic that is,

$$\exists \Lambda > 0 : \quad \langle A(x)\xi, \xi \rangle \geq \Lambda^{-1}|\xi|^2, \quad \forall \xi \in \mathbf{R}^d, \forall x \in B_{R_0}, \tag{1.1}$$

and such that,

$$A(0) = Id. \tag{1.2}$$

We set,

$$\mu(x) = \frac{\langle A(x)x, x \rangle}{|x|^2}. \tag{1.3}$$

It follows from the fact that A has Lipschitz entries that,

$$A(x) = Id + \mathcal{O}(|x|), \quad \mu(x) = 1 + \mathcal{O}(|x|), \quad \Lambda^{-1} \leq \mu(x) \leq \Lambda. \tag{1.4}$$

2. The doubling theorem

The main result of this section is the following.

Theorem 2.1. *Let $u \in H^1$ be a weak solution of the equation,*

$$Lu := div(A(x)\nabla_x u) = 0, \ \ in\ B_{R_0}$$

and let $R < \frac{R_0}{2}$. There exists $D > 0$ depending on R_0, u, d, Λ and on the Lipschitz constants of A such that, for every $r \in (0, R)$,

$$\int_{|x|<2r} |u(x)|^2 \, dx \leq D \int_{|x|<r} |u(x)|^2 \, dx. \tag{2.1}$$

Notice that every weak H^1 solution is C^α where $\alpha < 2$.

The rest of this section is devoted to the proof of this result.

Remark 2.2. If $u = P_n$ is a homogeneous harmonic polynomial of degree n, an exact computation shows that,

$$\int_{|x|<2r} |P_n(x)|^2 \, dx = 2^{2n+d} \int_{|x|<r} |P_n(x)|^2 \, dx.$$

2.1. **Preliminaries.** We set for $r \in (0, R)$,

$$H(r) = r^{1-d} \int_{|x|=r} \mu(x)|u(x)|^2 \, d\sigma_r, \tag{2.2}$$
$$I(r) = r^{1-d} \int_{|x|<r} \langle A(x)\nabla u(x), \nabla u(x)\rangle \, dx, \quad N(r) = \frac{rI(r)}{H(r)}.$$

($N(r)$ is called the frequency function.)

Lemma 2.3. *There exists $C > 0$ depending only on d, Λ and on the Lipschitz constants of A such that for any weak solution in H^1 of $Lu = 0$,*

$$(i) \quad N(r) = \frac{rH'(r)}{2H(r)} + \mathcal{O}(r), \qquad (ii) \quad \textit{the function } r \mapsto e^{Cr}N(r) \textit{ is non decreasing.}$$

Proof: Lemma 2.3 $\Longrightarrow$ Theorem 2.1

It follows from (i) that $H'(r) \leq 2r^{-1}N(r)H(r) + CH(r)$. On the other hand (ii) shows that for $r \leq R$ we have, $e^{Cr}N(r) \leq e^{CR}N(R)$ so $N(r) \leq e^{C(R-r)}N(R)$ and,

$$H'(r) \leq (2r^{-1}N(R)e^{C(R-r)} + C)H(r).$$

Let us integrate $\frac{H'}{H}$ on $[\rho, 2\rho]$. We obtain, for $2\rho \leq R$,

$$\mathrm{Log}H(2\rho) \leq \mathrm{Log}H(\rho) + 2N(R)e^{CR} \int_\rho^{2\rho} \frac{dr}{r} \leq (e^{CR}\mathrm{Log}4)N(R)$$

so,

$$H(2\rho) \leq \exp\big(e^{CR}(\mathrm{Log}4)N(R)\big)H(\rho).$$

It follows from the definition of H that,

$$(2\rho)^{1-d} \int_{|x|=2\rho} |u(x)|^2 \, d\sigma_{2\rho} \leq \exp\big(e^{CR}(\mathrm{Log}4)N(R)\big)\rho^{1-d} \int_{|x|=\rho} |u(x)|^2 \, d\sigma_\rho.$$

Dividing both members byρ^{1-d}, then integrating the inequality between 0 and r and using the fact that $\Lambda^{-1} \leq \mu(x) \leq M$ we obtain for $2r < R_0$,

$$\int_{|x|<2r} |u(x)|^2 \, dx \leq C(d, \Lambda, M)\exp\big(e^{CR}(\mathrm{Log}4)N(R)\big) \int_{|x|<r} |u(x)|^2 \, dx. \tag{2.3}$$

Remark 2.4. The constant D in (2.1) is of the form $C(d, \Lambda, M)\exp\big(e^{CR}(\mathrm{Log}4)N(R)\big)$. It depends on u through the exponential of the frequency function $N(R)$. It follows that the quantity $\frac{\|u\|_{L^2(B_{2r})}}{\|u\|_{L^2(B_r)}}$ is bounded by $C_1(d, \Lambda, M)\exp\big(e^{CR}(\mathrm{Log}2)N(R)\big)$. Therefore,

$$\mathrm{Log}\Big(\frac{\|u\|_{L^2(B_{2r})}}{\|u\|_{L^2(B_r)}}\Big) \leq C_2(d, \Lambda, M) + C_3(R)N(R).$$

This remark will be useful later on.

Proof of Lemma 2.3. We have first,

$$H(r) = r^{-d} \int_{|x|<r} \mathrm{div}\big(|u(x)|^2 A(x)x\big) \, dx. \tag{2.4}$$

This is a consequence of the divergence Theorem. Indeed the unit exterior normal to the ball beeing $\frac{x}{|x|}$ the integral of the right hand side is equal to,

$$\int_{|x|=r} |u(x)|^2 \langle A(x)x, \frac{x}{|x|}\rangle \, d\sigma_r = r \int_{|x|=r} |u(x)|^2 \frac{\langle A(x)x, x\rangle}{|x|^2} \, d\sigma_r = r^d H(r).$$

Let us compute the derivative of H. Using (2.4) we have,

$$H(r) = r^{-d} \int_0^r \int_{|x|=t} \operatorname{div}\big(|u(x)|^2 A(x)x\big) \, dt \, d\sigma_t,$$

therefore,

$$H'(r) = -dr^{-d-1} \int_{|x|<r} \operatorname{div}\big(|u(x)|^2 A(x)x\big) \, dx + r^{-d} \int_{|x|=r} \operatorname{div}\big(|u(x)|^2 A(x)x\big) \, d\sigma_r,$$

that means,

$$H'(r) = -dr^{-1} H(r) + r^{-d} \int_{|x|=r} \operatorname{div}\big(|u(x)|^2 A(x)x\big) \, d\sigma_r. \tag{2.5}$$

Consider the integral in the right hand side. We have,

$$\begin{aligned}
(1) = \operatorname{div}\big(|u(x)|^2 A(x)x\big) &= \sum_{j=1}^d \partial_j \big(|u(x)|^2 \sum_{k=1}^d a_{jk}(x)x_k\big),\\
&= \sum_{j=1}^d \big(2u(x)\partial_j u(x)\big(A(x)x\big)_j + |u(x)|^2 \sum_{k=1}^d (\partial_j a_{jk}(x))x_k + |u(x)|^2 a_{jj}(x)\big),\\
&= 2u(x)\langle A(x)x, \nabla u(x)\rangle + |u(x)|^2 A_D(x) + |u(x)|^2 \mathrm{Tr}A(x),
\end{aligned}$$

where $A_D(x) = \sum_{j,k=1}^d (\partial_j a_{jk}(x))x_k$. We have,

$$A_D(x) = \mathcal{O}(|x|), \quad \mathrm{Tr}A(x) = d + \mathcal{O}(|x|) = d\mu(x) + \mathcal{O}(|x|).$$

It follows that,

$$\begin{aligned}
\int_{|x|=r} \operatorname{div}\big(|u(x)|^2 A(x)x\big) \, dx = 2 \int_{|x|=r} u(x)\langle A(x)x, \nabla u(x)\rangle \, d\sigma_r + d \int_{|x|=r} \mu(x)|u(x)|^2 \, dx\\
+ \mathcal{O}\Big(r \int_{|x|=r} \mu(x)|u(x)|^2 \, dx\Big).
\end{aligned}$$

Therefore,

$$\int_{|x|=r} \operatorname{div}\big(|u(x)|^2 A(x)x\big) \, dx = 2 \int_{|x|=r} u(x)\langle A(x)x, \nabla u(x)\rangle \, d\sigma_r + dr^{d-1} H(r) + \mathcal{O}(r^d H(r)). \tag{2.6}$$

Since $A(x)$ is symmetric we have,

$$\int_{|x|=r} u(x)\langle A(x)x, \nabla u(x)\rangle\, d\sigma_r = \int_{|x|=r} u(x)\langle A(x)\nabla u(x), x\rangle\, d\sigma_r,$$
$$= r\int_{|x|=r} u(x)\langle A(x)\nabla u(x), \frac{x}{|x|}\rangle\, d\sigma_r = r\int_{|x|<r} \mathrm{div}\big(u(x)A(x)\nabla u(x)\big)\, dx,$$
$$= r\int_{|x|<r} \langle A(x)\nabla u(x), \nabla u(x)\rangle\, dx + r\int_{|x|<r} u(x)\mathrm{div}\big(A(x)\nabla u(x)\big)\, dx,$$
$$= r\int_{|x|<r} \langle A(x)\nabla u(x), \nabla u(x)\rangle\, dx,$$

since $Lu = 0$. Therefore,

$$\int_{|x|=r} u(x)\langle A(x)x, \nabla u(x)\rangle\, d\sigma_r = r\int_{|x|<r} \langle A(x)\nabla u(x), \nabla u(x)\rangle\, dx. \tag{2.7}$$

It follows from (2.6) that,

$$\int_{|x|=r} \mathrm{div}\big(|u(x)|^2 A(x)x\big)\, dx = 2r\int_{|x|<r} \langle A(x)\nabla u(x), \nabla u(x)\rangle\, dx + dr^{d-1}H(r) + \mathcal{O}(r^d H(r)).$$

We deduce from (2.5) that,

$$H'(r) = -dr^{-1}H(r) + 2r^{1-d}\int_{|x|<r} \langle A(x)\nabla u(x), \nabla u(x)\rangle\, dx + dr^{-1}H(r) + \mathcal{O}(H(r)),$$
$$= 2r^{1-d}\int_{|x|<r} \langle A(x)\nabla u(x), \nabla u(x)\rangle\, dx + \mathcal{O}(H(r)).$$

According to the definition of $I(r)$ en (2.2) we see that (i) is proved. Notice that from (2.7) we have,

$$I(r) = r^{-d}\int_{|x|=r} u(x)\langle A(x)x, \nabla u(x)\rangle\, d\sigma_r. \tag{2.8}$$

By (i) we have $rI(r) = \frac{1}{2}rH'(r) + \mathcal{O}(rH(r))$. therefore,

$$N(r) = \frac{rH'(r)}{2H(r)} + \mathcal{O}(r), \quad N'(r) = \frac{(rI(r))'}{H(r)} - \frac{rI(r)H'(r)}{H(r)^2}. \tag{2.9}$$

Let us show (ii). We compute $(rI(r))'$. From (2.2) we have,

$$rI(r) = r^{2-d}\int_0^r \int_{|x|=t} \langle A(x)\nabla u(x), \nabla u(x)\rangle d\sigma_t\, dt$$

so,

$$(rI(r))' = (2-d)I(r) + r^{2-d}\int_{|x|=r} \langle A(x)\nabla u(x), \nabla u(x)\rangle\, d\sigma_r \tag{2.10}$$

Let w be a vector field such that $\langle w, x\rangle = r^2$ on $|x| = r$. Then,

$$r^{1-d}\int_{|x|<r} \mathrm{div}(w(x)\langle A(x)\nabla u(x), \nabla u(x)\rangle)\, dx = r^{1-d}\int_{|x|=r} \langle A(x)\nabla u(x), \nabla u(x)\rangle\langle w, \nu\rangle\, d\sigma_r.$$

Since $\nu = \frac{x}{|x|}$ we have $\langle w, \nu\rangle = r$ so,

$$r^{1-d}\int_{|x|<r} \operatorname{div}(w(x)\langle A(x)\nabla u(x), \nabla u(x)\rangle)\, dx = r^{2-d}\int_{|x|=r} \langle A(x)\nabla u(x), \nabla u(x)\rangle\, d\sigma_r.$$

It follows that,

$$(rI(r))' = (2-d)I(r) + r^{1-d}\int_{|x|<r} \operatorname{div}(w(x)\langle A(x)\nabla u(x), \nabla u(x)\rangle)\, dx.$$

Therefore,

$$\begin{aligned}
&(rI(r))' = (2-d)I(r) + (1) + (2),\\
&(1) = r^{1-d}\int_{|x|<r} U(x)\, dx, \quad U(x) = \operatorname{div}(w)(x)\langle A(x)\nabla u(x), \nabla u(x)\rangle),\\
&(2) = r^{1-d}\int_{|x|<r} V(x)\, dx, \quad V(x) = w(x)\cdot\nabla\langle A(x)\nabla u(x), \nabla u(x)\rangle).
\end{aligned} \tag{2.11}$$

Let us compute the term $V(x)$. We can write,

$$\begin{aligned}
V(x) &= \sum_{j=1}^d w_j(x)\sum_{p,q=1}^d \partial_j a_{pq}(x)\partial_p u(x)\partial_q u(x) + 2\sum_{j=1}^d w_j(x)\sum_{p,q=1}^d a_{pq}(x)\partial_j\partial_p u(x)\partial_q u(x)\\
&= V_1(x) + V_2(x).
\end{aligned}$$

We have,

$$V_1(x) = \langle A_{Dw}(x)\nabla u(x), \nabla u(x)\rangle, \quad A_{Dw} = \big(\sum_{j=1}^d w_j\partial_j a_{pq}\big)_{1\le p,q\le d}. \tag{2.12}$$

Now $A\nabla u = \big(\sum_{q=1}^d a_{pq}\partial_p u\big)_{1\le p\le d}$, $\quad \operatorname{Hess}(u) = \big(\partial_p\partial_q u\big)_{1\le p,q\le d}$ so,

$$\big(\operatorname{Hess}(u)A\nabla u\big)_j = \sum_{p=1}^d \partial_j\partial_p u\sum_{q=1}^d a_{pq}\partial_q u = \sum_{p,q=1}^d a_{pq}\partial_j\partial_p u\partial_q u.$$

It follows that,

$$V_2(x) = 2\langle w(x), \operatorname{Hess}(u)(x)A(x)\nabla u(x)\rangle = 2\langle \operatorname{Hess}(u)(x)w(x), A(x)\nabla u(x)\rangle.$$

We are going to simplify the term V_2. We have,

$$\begin{aligned}
\langle\nabla\langle\nabla u, w\rangle, A\nabla u\rangle &= \sum_{j=1}^d \partial_j\big(\sum_{k=1}^d w_k\partial_k u\big)\big(A\nabla u\big)_j\\
&= \sum_{j,k=1}^d \partial_j w_k\partial_k u\big(A\nabla u\big)_j + \sum_{j,k=1}^d w_k\partial_j\partial_k u\big(A\nabla u\big)_j\\
&= \langle\langle Dw, \nabla u\rangle, A\nabla u\rangle + \langle \operatorname{Hess}(u)w, A\nabla u\rangle,
\end{aligned}$$

so,

$$V_2(x) = 2\langle\nabla\langle\nabla u, w\rangle, A\nabla u\rangle - 2\langle\langle Dw, \nabla u\rangle, A\nabla u\rangle.$$

Then,

$$\operatorname{div}\big(\langle\nabla u, w\rangle A\nabla u\big) = \langle\nabla\langle\nabla u, w\rangle, A\nabla u\rangle + \langle\nabla u, w\rangle\operatorname{div}\big(A\nabla u\big) = \langle\nabla\langle\nabla u, w\rangle, A\nabla u\rangle,$$

since $Lu=0$. It follows that,

$$V_2(x) = 2\mathrm{div}\big(\langle \nabla u, w\rangle A\nabla u\big) - 2\langle\langle Dw, \nabla u\rangle, A\nabla u\rangle.$$

Now from the Gauss-Green formula, since $\nu = \frac{x}{|x|}$, we have,

$$\int_{|x|<r} V_2(x)\,dx = 2r^{-1}\int_{|x|=r}\langle \nabla u, w\rangle\langle A\nabla u, x\rangle\,d\sigma_r - 2\int_{|x|<r}\langle\langle Dw, \nabla u\rangle, A\nabla u\rangle. \tag{2.13}$$

It follows from (2.11), (2.12), (2.13) that,

$$\begin{aligned}(2) = r^{1-d}\int_{|x|<r}\langle A_{Dw}(x)\nabla u(x), \nabla u(x)\rangle\,dx - 2r^{1-d}\int_{|x|<r}\langle\langle Dw, \nabla u\rangle, A\nabla u\rangle \\ + 2r^{-d}\int_{|x|=r}\langle \nabla u, w\rangle\langle A\nabla u, x\rangle\,d\sigma_r,\end{aligned}$$

so,

$$\begin{aligned}(rI(r))' &= (2-d)I(r) + r^{1-d}\int_{|x|<r}\mathrm{div}(w)(x)\langle A(x)\nabla u(x), \nabla u(x)\rangle) \\ &+ r^{1-d}\int_{|x|<r}\langle A_{Dw}(x)\nabla u(x), \nabla u(x)\rangle\,dx - 2r^{1-d}\int_{|x|<r}\langle\langle Dw(x), \nabla u(x)\rangle, A(x)\nabla u(x)\rangle \\ &+ 2r^{-d}\int_{|x|=r}\langle \nabla u(x), w(x)\rangle\langle A(x)\nabla u(x), x\rangle\,d\sigma_r = (2-d)I(r) + \sum_{k=1}^{4} J_k.\end{aligned}$$

We take $w(x) = \mu(x)^{-1}A(x)x$. It satisfies,

$$w(x) = \mathcal{O}(|x|) \quad \text{et} \quad \langle w(x), x\rangle = \frac{|x|^2}{\langle A(x)x, x\rangle}\langle A(x)x, x\rangle = |x|^2 = r^2, \quad \text{if } |x| = r.$$

We have $A(x) = Id + \mathcal{O}(|x|)$ so $A(x)x = x + \mathcal{O}(|x|^2)$, $\langle A(x)x, x\rangle = |x|^2 + \mathcal{O}(|x|^3)$, so, $w(x) = \frac{|x|^2}{|x|^2+\mathcal{O}(|x|^3)}(x + \mathcal{O}(|x|^2)) = x + \mathcal{O}(|x|^2)$. Then,

$$\begin{aligned}&Dw(x) = Id + \mathcal{O}(|x|), \quad \mathrm{div}w(x) = d + \mathcal{O}(|x|), \quad A_{Dw} = \mathcal{O}(|x|), \\ &\langle \nabla u(x), w(x)\rangle = \mu(x)^{-1}\langle A(x)\nabla u(x), x\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}J_1 &= dr^{1-d}\int_{|x|<r}\langle A(x)\nabla u(x), \nabla u(x)\rangle)\,dx + r^{1-d}\int_{|x|<r}\mathcal{O}(|x|)\langle A(x)\nabla u(x), \nabla u(x)\rangle)\,dx, \\ J_2 &\le Cr^{2-d}\int_{|x|<r}|\nabla u(x)|^2\,dx, \\ J_3 &= -2r^{1-d}\int_{|x|<r}\langle \nabla u(x), A(x)\nabla u(x)\rangle + r^{1-d}\int_{|x|<r}\langle \mathcal{O}(|x|)\nabla u(x), A(x)\nabla u(x)\rangle, \\ J_4 &= 2r^{-d}\int_{|x|=r}\mu(x)^{-1}\big(\langle A(x)\nabla u(x), x\rangle\big)^2\,d\sigma_r.\end{aligned}$$

According to (2.2) we have,

$$J_1 = dI(r) + \mathcal{O}(I(r)), \quad J_2 = \mathcal{O}(rI(r)), \quad J_3 = -2I(r) + \mathcal{O}(rI(r)),$$
$$J_4 = 2r^{-d} \int_{|x|=r} \big(\langle A(x)\nabla u(x), x\rangle\big)^2 \, d\sigma_r.$$

It follows that,

$$(rI(r))' = (2-d)I(r) + (d-2)I(r) + 2r^{-d} \int_{|x|=r} \big(\langle A(x)\nabla u(x), x\rangle\big)^2 \, d\sigma_r + \mathcal{O}(rI(r)),$$

so,

$$(rI(r))' = 2r^{-d} \int_{|x|=r} \big(\langle A(x)\nabla u(x), x\rangle\big)^2 \, d\sigma_r + \mathcal{O}(rI(r)). \tag{2.14}$$

Recall that,

$$N(r) = \frac{rI(r)}{H(r)}, \quad N'(r) = \frac{(rI(r))'}{H(r)} - \frac{rI(r)H'(r)}{H(r)^2}, \quad H'(r) = 2I(r) + \mathcal{O}(H(r)).$$

Then,

$$\begin{aligned}
\frac{N'(r)}{N(r)} &= \frac{(rI(r))'}{rI(r)} - \frac{H'(r)}{H(r)} = \frac{1}{rI(r)H(r)}\big((rI(r))'H(r) - rI(r)H'(r)\big), \\
&= \frac{1}{rI(r)H(r)}\big((rI(r))'H(r) - 2r\big(I(r)\big)^2 + \mathcal{O}(rI(r)H(r)\big), \\
&= \frac{1}{rI(r)H(r)}\big((rI(r))'H(r) - 2r\big(I(r)\big)^2\big) + \mathcal{O}(1).
\end{aligned}$$

Now from (2.8), (2.14) and the Hölder inequality we have,

$$\begin{aligned}
2rI(r)^2 &\le 2r^{1-2d}\Big(\int_{|x|=r} \mu(x)|u(x)|^2 \, d\sigma_r\Big)\Big(\int_{|x|=r} \mu(x)^{-1}\langle A(x)x, \nabla u(x)\rangle^2 \, d\sigma_r\Big), \\
&\le (rI(r))'H(r) + \mathcal{O}(rI(r)H(r)).
\end{aligned}$$

We deduce eventually that,

$$\frac{N'(r)}{N(r)} \ge \mathcal{O}(1),$$

in other words, there exists $C > 0$ such that $\frac{N'(r)}{N(r)} \ge -C$. Then, $\frac{d}{dr}\big(e^{Cr}N(r)\big) \ge 0$, which proves (ii) in Lemma 2.3.

□

3. The three-sphere theorem for elliptic operators.

Theorem 3.1. *Let $L = div(A\nabla)$ where A is a uniformly elliptic symmetric matrix with Lipschitz entries in a domain $\Omega \subset \mathbf{R}^d$. We assume that $B(0,4R) \subset \Omega$ and $A(0) = Id$. Then, for every $r < R$ there exists $\alpha \in (0,1), C > 0$ such that for every smooth solution of $Lu = 0$ in Ω we have,*

$$\int_{|x|=2r} |u(x)|^2 \, d\sigma_{2r} \le C\Big(\int_{|x|=r} |u(x)|^2 \, d\sigma_r\Big)^{\alpha}\Big(\int_{|x|=4r} |u(x)|^2 \, d\sigma_{4r}\Big)^{1-\alpha}.$$

Proof. By Lemma 2.3 we have $e^{Cr}N(r) \leq e^{2Cr}N(2r)$ that is, $N(r) \leq e^{Cr}N(2r)$ and $N(r) = \frac{rH'(r)}{2H(r)} + \mathcal{O}(1)$. Combining these two facts we get,

$$\frac{rH'(r)}{2H(r)} \leq e^{Cr}\Big(\frac{2rH'(2r)}{2H(2r)} + K\Big)$$

so,

$$\begin{aligned}\int_r^{2r} \frac{H'(\rho)}{H(\rho)}\,d\rho &\leq \int_r^{2r} e^{C\rho}\frac{2H'(2\rho)}{H(2\rho)}\,d\rho + 2K\int_r^{2r}\frac{e^{C\rho}}{\rho}\,d\rho,\\ &\leq e^{2Cr}\Big(\int_r^{2r}\frac{2H'(2\rho)}{H(2\rho)}\,d\rho + K' + 2K\int_r^{2r}\frac{d\rho}{\rho}\Big)\end{aligned}$$

since by Lemma 2.3 and the fact that $N(r) \geq 0$ there exists $K_0 > 0$ such that $\frac{H'(r)}{H(r)} + K_0 \geq 0$. It suffices to add (then substract) $2K_0$ in the integral to obtain a positive quantity.

Performing the integrations we get,

$$\mathrm{Log}H(2r) - \mathrm{Log}H(r) \leq e^{2Cr}\big(\mathrm{Log}H(4r) - \mathrm{Log}H(2r) + 2(\mathrm{Log}2)K\big),$$

so,

$$(1 + e^{2Cr})\mathrm{Log}H(2r) \leq \mathrm{Log}H(r) + e^{2Cr}\mathrm{Log}H(4r) + K''e^{2Cr},$$

which can be written, with $\alpha(r) = \frac{1}{1+e^{2Cr}}$,

$$\mathrm{Log}H(2r) \leq \mathrm{Log}(H(r))^{\alpha} + \mathrm{Log}(H(4r))^{1-\alpha} + K''(1-\alpha).$$

Taking the exponential of both members, using the definition of $H(\rho)$ and the fact that $\Lambda^{-1} \leq \mu(x) \leq \Lambda$ we obtain the theorem. ☐

Corollary 3.2. *Under the hypotheses of Theorem 3.1, for all $r < R$ there exists $\alpha \in (0,1)$ and $C > 0$ such that for any smooth solutions of $Lu = 0$ in Ω we have,*

$$\sup_{B_{2r}}|u| \leq C\big(\sup_{B_r}|u|\big)^{\alpha}(\sup_{B_{8r}}|u|)^{1-\alpha}.$$

Proof. By Corollary 6.8 we have,

$$\begin{aligned}(1) := (\sup_{|x|<2r}|u|)^2 &\leq C_1 r^{-d}\int_{|x|<4r}|u|^2\,dx \leq C_1 r^{-d}\int_0^{4r}\int_{|x|=\rho}|u|^2\,d\sigma_\rho\,d\rho,\\ &\leq C_2 r^{-d}\int_0^{r}\int_{|x|=4\rho}|u|^2\,d\sigma_{4\rho}\,d\rho\end{aligned}$$

Set,

$$s(\rho) = \int_{|x|=\rho}|u|^2\,d\sigma_\rho, \quad m(t) = (\sup_{|x|<t}|u|)^2$$

Using Theorem 3.1 with 2ρ then with ρ we get,

$$\begin{aligned}m(2r) &\leq C_3 r^{-d}\int_0^r s(2\rho)^{\alpha(2\rho)}s(8\rho)^{1-\alpha(2\rho)}\,d\rho,\\ &\leq C_4 r^{-d}\int_0^r\Big[s(\rho)^{\alpha(\rho)}s(4\rho)^{1-\alpha(\rho)}\Big]^{\alpha(2\rho)}s(8\rho)^{1-\alpha(2\rho)}d\rho.\end{aligned}$$

By the maximum principle we can write,

$$s(t) \leq C_5 t^{d-1}(\sup_{|x|=t}|u|)^2 \leq C_5 t^{d-1}(\sup_{|x|<t}|u|)^2 = C_5 t^{d-1}m(t)$$

so setting $\alpha_1(\rho) = \alpha(\rho)\alpha(2\rho)$ and bounding $m(4\rho)$ by $m(8\rho)$, we obtain,

$$m(2r) \le C_6 r^{-d} \int_0^r \rho^{d-1} m(\rho)^{\alpha_1(\rho)} m(8\rho)^{1-\alpha_1(\rho)}\, d\rho.$$

Since $\alpha(\rho) = \frac{1}{1+e^{2C\rho}}$ the function α_1 est decreasing. So $\alpha_1(r) \le \alpha_1(\rho)$ and since $m(\rho) \le m(8\rho)$ we get,

$$\Big(\frac{m(\rho)}{m(8\rho)}\Big)^{\alpha_1(\rho)} \le \Big(\frac{m(\rho)}{m(8\rho)}\Big)^{\alpha_1(r)}.$$

It follows that,

$$\begin{aligned} m(2r) &\le C_6 r^{-d} \int_0^r \rho^{d-1} m(\rho)^{\alpha_1(r)} m(8\rho)^{1-\alpha_1(r)}\, d\rho, \\ &\le C_6 r^{-d} \Big(\int_0^r \rho^{d-1}\, d\rho\Big) m(r)^{\alpha_1(r)} m(8r)^{1-\alpha_1(r)} \le C_7 m(r)^{\alpha_1(r)} m(8r)^{1-\alpha_1(r)}. \end{aligned}$$

□

Corollary 3.3. *There exist $r_0 > 0, k$ large enough, $C > 0, \alpha \in (0,1)$ such that if $B = B_r$ is a ball with $r < R_0$ and $B_{kr} = kB_r \subset \Omega$ we have,*

$$\sup_{B_{2r}} |u| \le C\big(\sup_{B_r} |u|\big)^{\alpha} (\sup_{B_{kr}} |u|)^{1-\alpha}.$$

Corollary 3.4. *Let $B \subset K \subset \Omega' \subset \Omega$ where B, Ω' are open, K is compact and $\overline{\Omega'} \subset \Omega$. There exists $\alpha \in (0,1), C > 0$ depending only on B, K, Ω', L, d such that for any continuous solution u in Ω of $Lu = 0$ we have,*

$$\sup_K |u| \le C\big(\sup_B |u|\big)^{\alpha}\big(\sup_{\Omega'} |u|\big)^{1-\alpha}.$$

Proof. Assume $\sup_{\Omega'} |u| = 1$. Fix a point $m_0 \in B$. For any $x \in K$ there is a curve connecting x to m_0. Then there exists a finite sequence of balls $(B_j)_{j=1}^J$ with radius $< r_0$ such that $B_1 \subset B$, $B_{j+1} \subset 2B_j$, $kB_j \subset \Omega'$ and $x \in B_J = B(x)$. Applying Corollary 3.3 we see that,

$$\sup_{B_{j+1}} |u| \le \sup_{2B_j} |u| \le C\big(\sup_{B_j} |u|\big)^{\alpha}.$$

Iterating this estimate we obtain,

$$\sup_{B_J} |u| \le C_J\big(\sup_B |u|\big)^{\alpha^J}, \alpha^J \in (0,1).$$

Eventually we use the fact that K can be covered by a finite number of ball $B(x)$. □

4. The doubling index

Let $u \in C^0(\Omega)$ be such that it does not vanish identically on any open subset of Ω. For any open ball B such that $2\overline{B}$ (the closed ball of same center and double radius) is contained in Ω we set,

$$N_u(B) = \operatorname{Log}\Big(\frac{\sup_{2B} |u|}{\sup_B |u|}\Big). \tag{4.1}$$

Example 4.1. Assume that P is a homogeneous polynomial of degree n and that $B = B(0,R)$. We have $\sup_{2B}|P| = \sup_{|x|\le 2R}(|x|^n \sum_{|\alpha|=n} a_\alpha \omega^\alpha) = (2R)^n c_n(\omega)$ and $\sup_B |P| = R^n c_n(\omega)$ so, $N_P(B) = \mathrm{Log}\, 2^n = n\mathrm{Log}\, 2$.

Let us compute the frequency function $N(r)$ of an harmonic polynomial. In that case we have $\mu(x) = 1$ and $H(r) = r^{1-d}\int_{|x|=r}|P(x)|^2\, d\sigma_r$. On the other hand, by (2.8),

$$I(r) = r^{-d}\int_{|x|=r} P(x) x\cdot\nabla P\, d\sigma_r = nr^{-d}\int_{|x|=r}|P(x)|^2\, d\sigma_r$$

since, P beeing homogeneous of degree n, the Euler relation shows that, $x\cdot\nabla P = nP$. Then, $N(r) = \frac{rI(r)}{H(r)} = n$.

In the general case we have we have the following result.

Lemma 4.2. *Let $B_r = \{x : |x| < r\}$ and let u be a continuous bounded solution of $Lu = 0$ in B_R.*

(*i*) *There exists $C_1 > 0$ such that , $N_u(B_r) \le C\big(N(R)+1\big)$, if $4r \le R$.*

(*ii*) *There exists $C_2 > 0$ such that, $N(r) \le C_2\big(N_u(B_r)+1\big)$.*

(*iii*) *There exists $C_3 > 0$ such that, $N_u(B_r) \le C_3(N_u(B_R)+1)$, if $4r \le R$.*

Proof. (*i*) There exists (see Corollary 6.8) $C \ge 1$ depending only on d,Λ such that,

$$C^{-1}t^{-\frac{d}{2}}\|v\|_{L^2(B_t)} \le \|v\|_{L^\infty(B_t)} \le Ct^{-\frac{d}{2}}\|v\|_{L^2(B_{2t})}. \tag{4.2}$$

Applying these inequalities with $t = 2r$ and $t = r$ we obtain,

$$\mathrm{Log}\Big(\frac{\|u\|_{L^\infty(B_{2r})}}{\|u\|_{L^\infty(B_r)}}\Big) \le \mathrm{Log}\Big(C_0\frac{\|u\|_{L^2(B_{4r})}}{\|u\|_{L^2(B_r)}}\Big) = \mathrm{Log}\Big(\frac{\|u\|_{L^2(B_{4r})}}{\|u\|_{L^2(B_{2r})}}\Big) + \mathrm{Log}\Big(\frac{\|u\|_{L^2(B_{2r})}}{\|u\|_{L^2(B_r)}}\Big) + C_1$$

From the inequality (2.3) we have,

$$\int_{|x|<2t}|u(x)|^2\, dx \le C_2 e^{C_2 N(T)}\int_{|x|<t}|u(x)|^2\, dx, \tag{4.3}$$

where $2t \le T$ and C_2 depends only on d,Λ and the Lipschitz constants of A. Apply this inequality with $t = 2r, t = r$ and $T = R$. We get,

$$\mathrm{Log}\Big(\frac{\|u\|_{L^2(B_{4r})}}{\|u\|_{L^2(B_{2r})}}\Big) + \mathrm{Log}\Big(\frac{\|u\|_{L^2(B_{2r})}}{\|u\|_{L^2(B_r)}}\Big) \le C_3 N(R) + C_4.$$

$$N_u(B_r) = \mathrm{Log}\Big(\frac{\|u\|_{L^\infty(B_{2r})}}{\|u\|_{L^\infty(B_r)}}\Big) \le C_4 N(R) + C_5,$$

which proves (*i*).

(*ii*) We use (*i*) in Lemma 2.3, that is,

$$N(r) = \frac{rH'(r)}{2H(r)} + \mathcal{O}(r) \iff \frac{2}{r}N(r) = \frac{H'(r)}{H(r)} + \mathcal{O}(1).$$

We integrate this inequality between $\frac{3}{2}r$ and $2r$ where $0 < r < R_0$. We get,

$$\int_{\frac{3}{2}r}^{2r}\frac{2}{\rho}N(\rho)d\rho \le \mathrm{Log}\frac{H(2r)}{H(\frac{3}{2}r)} + Cr.$$

We use (see Lemma 2.3) the fact that the function $\rho \mapsto e^{C\rho}N(\rho)$ is non decreasing so, $N(\rho) \geq e^{C(r-\rho)}N(r) \geq e^{-CR_0}N(r)$. We deduce that,

$$2(\mathrm{Log}2)e^{-CR_0}N(r) \leq \mathrm{Log}\frac{H(2r)}{H(\frac{3}{2}r)} + CR_0. \tag{4.4}$$

Then we can write,

$$H(2r) \leq C(\sup_{|x|=2r}|u|)^2 \leq C(\sup_{|x|<2r}|u|)^2. \tag{4.5}$$

On the other hand, by Theorem 6.5 we have for $\rho > 0$,

$$\begin{aligned}\|u\|^2_{L^\infty(B_\rho)} &\leq C_1\rho^{-d}\int_{B_{\frac{3}{2}\rho}}|u|^2\,dx = C_1\rho^{-d}\int_0^{\frac{3}{2}\rho}t^{d-1}t^{1-d}\int_{|x|=t}|u(x)|^2\,d\sigma_t\,dt\\ &\leq C_2\sup_{0<t<\frac{3}{2}\rho}H(t).\end{aligned}$$

Now from (i) in Lemma 2.3 we have $\frac{H'}{H} \geq -C$ so the function $t \mapsto e^{Ct}H(t)$ is non decreasing. We deduce that,

$$\|u\|^2_{L^\infty(B_r)} \leq C_3e^{C'r}H(\frac{3}{2}r),$$

so,

$$\frac{1}{H(\frac{3}{2}r)} \leq \frac{C_4e^{C'r}}{\|u\|^2_{L^\infty(B_r)}}. \tag{4.6}$$

Using (4.5) we obtain,

$$\frac{H(2r)}{H(\frac{3}{2}r)} \leq C_5e^{C'r}\frac{\|u\|^2_{L^\infty(B_{2r})}}{\|u\|^2_{L^\infty(B_r)}}.$$

It follows that for $r \leq R_0$,

$$\mathrm{Log}\frac{H(2r)}{H(\frac{3}{2}r)} \leq C_6 + C_7R_0 + N_u(B_r).$$

We have just to use (4.4) to conclude.

(iii) Indeed from (i) we have $N_u(B_r) \leq C(N(R)+1)$ when $4r \leq R$ and from (ii) we have $N(R) \leq C\big(N_u(B_R)+1\big)$. □

4.1. **Doubling index of the eigenfunctions.** Let ϕ_λ be an eigenfunction of $-\Delta_g$ that is, $-\Delta_g\phi_\lambda = \lambda\phi_\lambda, \lambda \geq 0$. Then $h(t,x) = e^{t\sqrt{\lambda}}\phi_\lambda$ is a solution of $(\partial_t^2 + \Delta_g)h = 0$ and we can apply the previous results. We obtain a three-sphere inequality and a notion of doubling index. This has been used by Donnelly-Fefferman in their study of the nodal sets of the eigenfunctions. A result they used is the following.

Theorem 4.3. *Let (M,g) be a smooth compact Riemannian manifold without boundary. There exists $r_0, C > 0$ depending on M such that for any eigenfunction of $-\Delta_g$ corresponding to the eigenvalue λ we have,*

$$N_{\phi_\lambda}(B_r) \leq C(1+\sqrt{\lambda}).$$

This result suggests that the eigenfunctions of $-\Delta_g$ corresponding to the eigenvalue λ behave like polynomials of degree $\sqrt{\lambda}$.

Proof. Set $u(t,x) = e^{t\sqrt{\lambda}}\phi_\lambda$. Then u solves the equation $(\partial_t^2 + \Delta_g)u = 0$ on $\mathbf{R}\times M$ and we may apply the previous results to u. We may assume that e $\sup_M |\phi_\lambda| = |\phi_\lambda(\overline{x})| = 1$. let $r > 0$ be so small that for all $x \in M$ the geodesic ball of center x and radius r is contained in a chart. Let $k \in \mathbf{N}, k \geq 3$. Let B be a ball of radius $\frac{r}{2k}$ in M and $\widetilde{B} = (-\frac{r}{2k}, \frac{r}{2k}) \times B$. We choose a finite family of geodesic balls $(\widetilde{B}_j)_{j=1}^J$ centered at $(0, x_j)$ in $\mathbf{R}\times M$ of equal radius $\frac{r}{2k}$ such that,

$$\widetilde{B}_1 = \widetilde{B}, \quad \widetilde{B}_{j+1} \subset 2\widetilde{B}_j, \quad (0,\overline{x}) \in \widetilde{B}_J.$$

We apply Corollary 3.2. We get,

$$\sup_{\widetilde{B}_{j+1}} |u| \leq \sup_{2\widetilde{B}_j} |u| \leq C\big(\sup_{\widetilde{B}_j} |u|\big)^{\beta}\big(\sup_{k\widetilde{B}_j} |u|\big)^{1-\beta}.$$

Since $\widetilde{B}_j = (-\frac{r}{2k}, \frac{r}{2k}) \times B_j$ we have, $\sup_{\widetilde{B}_j} |u| = e^{\frac{r}{2k}\sqrt{\lambda}} \sup_{B_j} |\phi_\lambda|, \quad \sup_{k\widetilde{B}_j} |u| \leq e^{\frac{r}{2}\sqrt{\lambda}}$, since, $\sup_{kB_j} |\phi_\lambda| \leq \sup_M |\phi_\lambda| = 1$. We deduce that,

$$\sup_{B_j} |\phi_\lambda| \geq C_1 e^{-m\sqrt{\lambda}}\big(\sup_{B_{j+1}} |\phi_\lambda|\big)^{\frac{1}{\beta}},$$

where $m = \frac{1-\beta}{\beta}\big(\frac{r}{2} - \frac{r}{2k}\big)$. Therefore for all $j \geq 2$,

$$\sup_{B_1} |\phi_\lambda| \geq C^{k_j} e^{-m_j\sqrt{\lambda}}\big(\sup_{B_j} |\phi_\lambda|\big)^{\frac{1}{\beta^j}},$$

where $k_j = \sum_{\ell=0}^{j-2} \frac{1}{\beta^\ell}$, $m_j = m\sum_{\ell=0}^{j-2} \frac{1}{\beta^\ell}$. Taking $j = J$ and using the fact that $\sup_{B_J} |\phi_\lambda| = 1$ since $\overline{x} \in B_J$ we obtain,

$$\sup_{B} |\phi_\lambda| \geq C^{k_J} e^{-m_J\sqrt{\lambda}}.$$

Let now B_r be a ball of radius r which contains B and such that B_{2r} is contained in a chart. We have,

$$\sup_{B_r} |\phi_\lambda| \geq C^{k_J} e^{-m_J\sqrt{\lambda}}$$

so,

$$\frac{\sup_{B_{2r}} |\phi_\lambda|}{\sup_{B_r} |\phi_\lambda|} \leq \frac{\sup_M |\phi_\lambda|}{\sup_{B_r} |\phi_\lambda|} \leq \frac{1}{\sup_{B_r} |\phi_\lambda|} \leq C^{-k_J} e^{m_J\sqrt{\lambda}},$$

and,

$$N_{\phi_\lambda}(B_r) = \mathrm{Log}\Big(\frac{\sup_{2B_{2r}} |\phi_\lambda|}{\sup_{B_r} |\phi_\lambda|}\Big) \leq C(1+\sqrt{\lambda}).$$

□

4.2. **The doubling index on cubes.** If Q is a cube in $\mathbf{R}^d$ of length side $s(Q)$ we shall denote by λQ, for $\lambda > 0$, the cube of same center and length side $\lambda s(Q)$.

We define the doubling index $\mathcal{N}_u(Q)$ as follows,

$$\mathcal{N}_u(Q) = \sup_{q\subset Q} N_u(q), \quad N_u(q) = \mathrm{Log}\,\Big(\frac{\sup_{2q} |u|}{\sup_q |u|}\Big).$$

Proposition 4.4. *There exist positive constants a_1, a_2 depending only on Λ and on the Lipschitz constants of A such that for any cube $Q \subset \mathbf{R}^d$ with $s(Q) \leq 1$ and any bounded continuous solution u of $Lu = 0$ in $2Q$ we have,*

$$N_u(Q) \leq \mathcal{N}_u(Q) \leq a_1 N_u(Q) + a_2.$$

Proof. The left inequality is trivial. Let us prove the right one. Let q be a cube, $q \subset Q = \{x : |x_j - a_j| \leq \frac{1}{2}s(Q)\}$ where $s(Q)$ denotes the length of a side of Q.

Cas 1. $s(q) \leq c_d s(Q), c_d << 1$.

Let $b = b(x^0, \frac{1}{2}s(q))$ the biggest ball inscribed in q. Then, $2q \subset k_d b$ where $k_d = \sqrt{2d}$. Let $B = B(x^0, \frac{1}{4}s(Q))$. We have $2B \subset 2Q$ because if $x \in 2B$ we have,

$$|x_j - a_j| \leq |x_j - x_j^0| + |x_j^0 - a_j| \leq \frac{1}{2}s(Q) + \frac{1}{2}s(Q) = s(Q).$$

Now let $m = x^0 + \mu(a - x^0), \mu = \frac{1}{2(1+\sqrt{d})}$ and $B_0 = \{x : |x - m| \leq \frac{1}{2}\mu s(Q)\}$. Then $B_0 \subset Q$ et $B_0 \subset B$. Indeed if $x \in B_0$ we have,

$$|x_j - a_j| = |x_j - m_j + m_j - a_j| \leq |x_j - m_j| + (1-\mu)|x_j^0 - a_j| \leq \frac{1}{2}\mu s(Q) + (1-\mu)\frac{1}{2}s(Q) = \frac{1}{2}s(Q).$$

On the other hand,

$$|x - x^0| \leq |x - m| + \mu|a - x^0| \leq \frac{1}{2}\mu s(Q) + \mu\sqrt{d}\frac{1}{2}s(Q) = \frac{1}{2}\mu(1+\sqrt{d})s(Q) = \frac{1}{4}s(Q).$$

Let $\ell \in \mathbf{N}$ be such that $2^\ell \leq k_d < 2^{\ell+1}$. We write,

$$\frac{\sup_{2q}|u|}{\sup_q |u|} \leq \frac{\sup_{k_d b}|u|}{\sup_b |u|} = \prod_{j=1}^{\ell} \frac{\sup_{\frac{1}{2^{j-1}}k_d b}|u|}{\sup_{\frac{1}{2^j}k_d b}|u|} \times \frac{\sup_{\frac{1}{2^\ell}k_d b}|u|}{\sup_b |u|},$$
$$\leq \prod_{j=1}^{\ell} \frac{\sup_{\frac{1}{2^{j-1}}k_d b}|u|}{\sup_{\frac{1}{2^j}k_d b}|u|} \times \frac{\sup_{2b}|u|}{\sup_b |u|}.$$

It follows that,

$$\mathrm{Log}\Big(\frac{\sup_{2q}|u|}{\sup_q |u|}\Big) \leq \sum_{j=1}^{\ell} N_u\big(\frac{1}{2^{j-1}}k_d b\big) + N_u(b).$$

Since $s(q) << s(Q)$ we deduce from (iii) in Lemma 4.2 that,

$$\mathrm{Log}\Big(\frac{\sup_{2q}|u|}{\sup_q |u|}\Big) \leq C_d(N_u(B) + 1). \tag{4.7}$$

Now since the radius of B_0 is uniformly equivalent to $s(Q)$ by Corollary 3.4 there exist constants $A, \gamma \in (0,1)$ depending only on the dimension such that,

$$\sup_Q |u| \leq A\Big(\sup_{B_0}|u|\Big)^{\gamma}\Big(\sup_{2Q}|u|\Big)^{1-\gamma}.$$

It follows that,

$$\mathrm{Log}\Big(\frac{\sup_{2Q}|u|}{\sup_Q |u|}\Big) \geq A_1 \mathrm{Log}\Big(\frac{\sup_{2Q}|u|}{\sup_{B_0}|u|}\Big) - A_2 \geq A_1 \mathrm{Log}\Big(\frac{\sup_{2B}|u|}{\sup_B |u|}\Big) - A_2 = A_1 N_u(B) - A_2$$

since $2B \subset 2Q$ and $B_0 \subset B$. Using (4.7) we obtain,

$$\mathcal{N}_u(Q) = \sup_{q \subset Q} \operatorname{Log}\Big(\frac{\sup_{2q}|u|}{\sup_q |u|}\Big) \le a_1 \operatorname{Log}\Big(\frac{\sup_{2Q}|u|}{\sup_{2Q}|u|}\Big) + a_2$$

where a_j depend only on d.

Cas 2. $s(q) \ge c_d s(Q)$.

In that case we can use the three-sets theorem with $q \subset Q \subset 2Q$ and we obtain with constants depending only on the dimension (when $s(Q) \le 1$),

$$\sup_Q |u| \le C\big(\sup_q |u|\big)^{\gamma}\big(\sup_{2Q}|u|\big)^{1-\gamma},$$

which imply that,

$$\frac{\sup_{2Q}|u|}{\sup_Q |u|} \ge \frac{1}{C}\Big(\frac{\sup_{2Q}|u|}{\sup_q |u|}\Big)^{\gamma} \ge \frac{1}{C}\Big(\frac{\sup_{2q}|u|}{\sup_q |u|}\Big)^{\gamma},$$

so,

$$\mathcal{N}_u(Q) = \sup_{q \subset Q} \operatorname{Log}\Big(\frac{\sup_{2q}|u|}{\sup_q |u|}\Big) \le a_1 \operatorname{Log}\Big(\frac{\sup_{2Q}|u|}{\sup_{2Q}|u|}\Big) + a_2.$$

□

4.2.1. *A lemma on cubes.*

Lemma 4.5. *Let Q be a cube. We partition it into K^d equal cubes. Let q be one of the cubes of the partition. Then,*

(i) $Q \subset 2Kq \subset 3Q, \quad Kq \subset 2Q,$

(ii) If $q \cap \Big(\frac{1}{2} + \frac{3m}{K}\Big)Q \neq \emptyset$, *then* $2q \subset \Big(\frac{1}{2} + \frac{3m+1}{K}\Big)Q, \quad \forall m \in \mathbf{N},$

(iii) If $q \cap \frac{1}{2}Q \neq \emptyset$, *then* $\frac{1}{3}Kq \subset Q$.

Proof. See the appendix. □

5. Propagation of smallness for solutions of elliptic equations.

Let $L = \operatorname{div}(A\nabla)$ be a uniformly elliptic operator with Lipschitz coefficients in $\Omega \subset \mathbf{R}^d$. We know that a solution of the equation $Lh = 0$ which vanishes on a set of positive measure vanishes identically. The purpose of this section is give a quantitative version of this result.

Theorem 5.1. *Let h be a bounded continuous solution of $Lh = 0$ in Ω. Let $E \subset \Omega$ be measurable with strictly positive measure. Assume that $|h| \le \varepsilon$ on E. Let $K \subset \Omega$ be a compact subset. Then there exists $\alpha \in (0,1), C > 0$ depending only on $A, |E|, \operatorname{dist}(E, \partial\Omega), K, d$ such that,*

(5.1) $$\sup_K |h| \le C(\sup_E |h|)^{\alpha}\big(\sup_{\Omega}|h|\big)^{1-\alpha}.$$

The rest of this section is devoted to the proof of this result.

We begin by proving this theorem with $K = \overline{Q}$ where Q is the unit cube, $\Omega = 2Q$ and the equation holds in 3Q, that is,

(5.2) $$\sup_Q |h| \le C(\sup_E |h|)^{\alpha}\big(\sup_{2Q}|h|\big)^{1-\alpha}.$$

Remark 5.2. We notice that in the above theorem one may assume that $\sup_K |h| \geq 2\sup_E |h|$ otherwise (5.1) is trivial.

Indeed assume that $\sup_K |h| \leq 2\sup_E |h|$. Then,

$$\sup_K |h| \leq 2\sup_E |h| = 2(\sup_E |h|)^{\alpha}(\sup_E |h|)^{1-\alpha} \leq 2(\sup_E |h|)^{\alpha}(\sup_{\Omega} |h|)^{1-\alpha}$$

for every $\alpha \in (0,1)$.

5.0.1. *Preliminaries.* We first prove that the solutions h for which $\sup_E |h| \leq \frac{1}{2}\sup_Q |h|$ have a doubling index $N = \mathcal{N}_u(Q)$ bounded below. For this we shall use Corollary 6.13.

Lemma 5.3. *Let $M = \sup_Q |h|$. Assume that $\sup_E |h| \leq \frac{1}{2}M$. Let $\ell_0 \geq 2$ and $K_0 = 2^{\ell_0 - 1} \geq 1$ be such that $\tau(\frac{1}{K_0}) \leq \frac{1}{24}$. Then, $N \geq \frac{1}{\ell_0}Log(\frac{4}{3}) := n_0$.*

Proof. Let $\ell_0 \geq 2, K_0 = 2^{\ell_0-1}$ be such that $6\tau(\frac{1}{K_0}) \leq \frac{1}{4}$. Then we shall prove that, $N = \mathcal{N}_h(Q) \geq \frac{1}{\ell_0}\text{Log}\frac{4}{3}$.

If $N \geq 1$ we are done since, $1 \geq \frac{1}{\ell_0}\text{Log}\frac{4}{3}$. Assume $N \leq 1$.

Recall that $N = \mathcal{N}_h(Q) = \sup_{q\subset Q} \text{Log}\Big(\frac{\sup_{2q}|u|}{\sup_q |u|}\Big)$. Cut Q into K_0^d equal cubes and let q be one of them. By Lemma 4.5 we have,

$$Q \subset 2K_0 q = 2^{\ell_0} q \subset 3Q, \quad K_0 q \subset 2Q. \tag{5.3}$$

Then, using (5.3) and iterating we obtain,

$$M = \sup_Q |h| \leq \sup_{2^{\ell_0}q} |h| \leq e^{N\ell_0} \sup_q |h|, \qquad \sup_{2Q} |h| \leq e^N \sup_Q |h| \leq 3M. \tag{5.4}$$

By Corollary 6.13 with $s = \frac{1}{K_0}$, and the fact that $osc_{2Q}h \leq 2\sup_{2Q}|h| \leq 6M$,

$$\text{osc}_q u = \text{osc}_{\frac{1}{K_0}K_0 q} h \leq \tau(\frac{1}{K_0})\text{osc}_{K_0 q} h \leq \tau(\frac{1}{K_0})\text{osc}_{2Q} h \leq 6M\tau(\frac{1}{K_0}) \leq \frac{M}{4}. \tag{5.5}$$

We may assume that $\sup_q h = u(x_0) > 0$. Assume that $N < \frac{1}{\ell_0}\text{Log}(\frac{4}{3})$, then $e^{-N\ell_0} > \frac{3}{4}$. Therefore, since $M \geq 2\sup_E |h|$, for every cube q we have,

$$\inf_q h = \sup_q h - \text{osc}_q h \geq \big(e^{-N\ell_0} - \frac{1}{4}\big)M > \frac{1}{2}M \geq \sup_E |h|.$$

This is a contradiction Therefore $N \geq \frac{1}{\ell_0}\text{Log}(\frac{4}{3})$. □

5.1. Remez inequalities for solutions of elliptic equations.

5.1.1. *Introduction.* Initially the Remez inequalities concerned polynomials. The question was the following. If P_n is a polynomial of degree n and if we know that,

$$|\{x \in [-1,1] : |P_n(x)| \leq 1\}| \geq 2 - s, \quad 0 < s < 2,$$

(where $|\cdot|$ is the Lebesgue measure) can we have an estimate of the size of P_n on $[-1,1]$? This was answered by Remez as follows. Under these hypotheses we have,

$$\sup_{[-1,1]} |P_n| \leq T_n\Big(\frac{2+s}{2-s}\Big) \tag{5.6}$$

where T_n is the Tchebycheff polynomial of degree n and we have equality if and only if $P_n = \pm T_n\Big(\frac{\pm 2x + s}{2-s}\Big)$.

Here is a consequence of this result. Let E be a measurable subset of $[-1,1]$ then,

$$\sup_{[-1,1]} |P_n| \leq \Big(\frac{8}{|E|}\Big)^n \sup_E |P_n|. \tag{5.7}$$

The proof is the following. Take $s = 2 - |E| \in (0,2)$ and set $\widetilde{P}_n = \frac{P_n}{\sup_E |P_n|}$. Then,

$$E \subset \{x \in [-1,1] : |\widetilde{P}_n(x)| \leq 1\}.$$

Therefore,

$$|\{x \in [-1,1] : |\widetilde{P}_n(x)| \leq 1\}| \geq |E| = 2 - s.$$

We may apply the inequality (5.6) and deduce that,

$$\sup_{[-1,1]} |\widetilde{P}_n| \leq T_n\Big(\frac{4-|E|}{|E|}\Big).$$

Now for $x > 0$ large enough we have, $T_n(2x-1) \leq (4x)^n$ and $2x - 1 = \frac{4-|E|}{|E|} \iff x = \frac{2}{|E|}$. Therefore if $|E|$ is small enough we obtain (5.7).

According to the analogy made previously between polynomials of degree n and solutions of elliptic equations with doubling index N, A. Logunov and E. Malinnikova prove the following.

Theorem 5.4. *Let Q be the unit cube in $\mathbf{R}^d$ and let h be a bounded continuous solution of $Lh = 0$ in $2Q$. Set $N = \mathcal{N}_h(Q)$ the doubling index and assume that $N \geq n_0$. Then for every measurable set $E \subset Q$ with strictly positive measure $|E|$, we have,*

$$\sup_Q |h| \leq C \sup_E |h| \Big(C\frac{|Q|}{|E|}\Big)^{CN}, \tag{5.8}$$

where C depends only on d and A.

5.1.2. *Theorem 5.4 implies Theorem 5.1.* We have seen that,

$$\mathrm{Log}\, \frac{\sup_{2Q} |h|}{\sup_Q |h|} \geq a_1 \mathcal{N}_h(Q) - a_2, \quad a_j > 0.$$

We deduce that,

$$e^{a_1 N} \leq e^{a_2} \frac{\sup_{2Q} |h|}{\sup_Q |h|}. \tag{5.9}$$

Assume (5.8) true. Choose $C_1 = C_1(|E|)$ such that $\Big(C\frac{|Q|}{|E|}\Big)^C = e^{a_1 C_1}$ that is $C_1 = a_1^{-1} C\, \mathrm{Log}\, (C|Q||E|^{-1})$. Using (6.14) we obtain,

$$\sup_Q |h| \leq C e^{a_1 C_1 N} \sup_E |h| \leq C e^{a_2 C_1} \sup_E |h| \big(\sup_{2Q} |h|\big)^{C_1} \big(\sup_Q |h|\big)^{-C_1}$$

so,

$$\big(\sup_Q |h|\big)^{1+C_1} \leq C_2 \sup_E |h| \big(\sup_{2Q} |h|\big)^{C_1}$$

and eventually,

$$\sup_Q |h| \leq C_3 \big(\sup_E |h|\big)^{\frac{1}{1+C_1}} \big(\sup_{2Q} |h|\big)^{\frac{C_1}{1+C_1}},$$

which proves Theorem 5.1 with $\alpha = \frac{1}{1+C_1}$ in the case where $K = \overline{Q}, \Omega = 2Q$. The general case can be proved as in Corollary 3.4.

Here is an equivalent version of Theorem 5.4.

Theorem 5.5. *Let Q be the unit cube in $\mathbf{R}^d$ and let h be a bounded continuous solution of $Lh = 0$ in $2Q$ such that $\sup_Q |h| = 1$. Set $N = \mathcal{N}_h(Q) \geq n_0$ and,*

$$E_a(h) = \{x \in Q : |h(x)| \leq e^{-a}\}, \quad a > 0.$$

Then there exists $\beta > 0, C > 0$ depending only on A, d such that,

$$|E_a(h)| \leq Ce^{-\frac{\beta a}{N}}|Q|, \tag{5.10}$$

Let us show the equivalence of these two theorems.

(i) Theorem 5.4 implies Theorem 5.5.

By Theorem 5.4 we have $1 \leq C \sup_E |h| \left(C\frac{|Q|}{|E|}\right)^{CN}$. Take $E = E_a(h)$, then $\sup_E |h| \leq e^{-a}$ so that, $\left(\frac{|E_a(h)|}{C|Q|}\right)^{CN} \leq Ce^{-a}$ or, $|E_a(h)| \leq C|Q|\left(Ce^{-a}\right)^{\frac{1}{CN}}$. Since $N \geq n_0$ we have $C^{\frac{1}{CN}} \leq C^{\frac{1}{Cn_0}}$ so, $|E_a(h)| \leq C'|Q|e^{-\frac{\beta a}{N}}, \beta = \frac{1}{C}$.

(ii) Theorem 5.5 implies Theorem 5.4.

Let $|E| > 0$. We have, $\sup_E |h| \leq \sup_Q |h| = 1$. If $\sup_E |h| = \sup_Q |h|$ the inequality (5.8) is satisfied as soon as $C \geq 1$. If $\sup_E |h| < \sup_Q |h|$ there exists $a > 0$ such that $\sup_E |h| = e^{-a}$. Then $E \subset E_a(h)$ and $|E| \leq |E_a(h)| \leq Ce^{-\frac{\beta a}{N}}|Q|$ so, $1 \leq C\frac{|Q|}{|E|}e^{-\frac{\beta a}{N}}$. Taking the power $\frac{N}{\beta}$ of both members we obtain,

$$1 \leq e^{-a}\left(C\frac{|Q|}{|E|}\right)^{\frac{N}{\beta}} = \sup_E |h| \left(C\frac{|Q|}{|E|}\right)^{\frac{N}{\beta}},$$

which proves Theorem 5.4 if $\sup_Q |h| = 1$; the general case can be obtained considering $\frac{h}{\sup_Q |h|}$.

The rest of this section will be devoted to the proof of Theorem 5.5

5.2. **Beginning of the proof of Theorem 5.5.** We begin by proving the result in the case where $\frac{a}{N} \leq c_0$ and $N \leq N_0$. Then we shall make a double induction on a and N.

Cas 1. $\frac{a}{N} \leq c_0$

In that case $c_0 - \frac{a}{N} \geq 0$. Since $E_a(h) \subset Q$ we have,

$$|E_a(h)| \leq |Q| \leq e^{c_0 - \frac{a}{N}}|Q| = e^{c_0}e^{-\frac{a}{N}}|Q|.$$

Cas 2. $n_0 \leq N \leq N_0$.

Proposition 5.6. *let h be a bounded continuous solution of $Lh = 0$ in k_dQ with $\sup_Q |h| = 1$ and $\mathcal{N}_h(Q) \leq N_0$. Let $E_a(h) = \{x \in Q : |h(x)| \leq e^{-a}\}$. Then there exist positive constants γ, C depending only on A, d, N_0 such that,*

$$|E_a(h)| \leq Ce^{-\gamma a}|Q|.$$

Notice that since $N \geq n_0$ we have $-\gamma a \leq -\frac{(\gamma n_0)a}{N}$.

Proof. We have, $\sup_{q \in Q} \text{Log} \frac{\sup_{2q} |h|}{\sup_q |h|} \leq N_0$. In particular if $q = \frac{1}{2}Q$ and since $\sup_Q |h| = 1$ we have,

$$\sup_{\frac{1}{2}Q} |h| \geq e^{-N_0}.$$

We combine this with a result about the oscillations. For that we recall some facts..

Lemma 5.7. *Let Q be a cube and $\lambda \in (0,1)$. Assume $Lh = 0$ in $3Q$ with $\sup_Q |h| \geq \lambda$ and $\mathcal{N}_h(Q) \leq N_0$. There exists $K > 0, b \in (0,1), m_0 \in (0,1)$ depending on N_0, d, A but independent of λ such that if Q is cut into K^d equal cubes, $Q = \cup_i q_i$ then,*

(i) there exists a cube q_0 such that,

$$\inf_{q_0} |h| \geq m_0 \lambda, \tag{5.11}$$

(ii) for every cube q_i we have,

$$\sup_{q_i} |h| \geq b\lambda \quad \forall i = 1, \dots, K^d, \tag{5.12}$$

Proof. Since $\mathcal{N}_{\frac{h}{\lambda}}(Q) = \mathcal{N}_h(Q)$ and $L(\frac{1}{\lambda}h) = 0$ it is sufficient to prove the lemma with $\lambda = 1$ then to apply it to the function $\frac{1}{\lambda}h$.

Let us show (5.11). Firstly, by Lemma 4.5, if $q \cap \frac{1}{2}Q \neq \emptyset$ we have, $\frac{1}{3}Kq \subset Q$. Next writing $\frac{3}{K}\frac{K}{3}q = q$ we apply (6.12) with $Q = \frac{K}{3}q, s = \frac{3}{K}$. We get,

$$\text{osc}_q h \leq \tau(\frac{3}{K}) \text{osc}_{\frac{K}{3}q} h \leq \tau\big(\frac{3}{K}\big) \text{osc}_Q h$$

since the oscillation is a non decreasing function of the set. Now since $\mathcal{N}_h(Q) \leq N_0$ and $\sup_Q |h| = 1$ we have $\sup_{\frac{1}{2}Q} |h| \geq e^{-N_0}$. Let x_0 be such that $|h(x_0)| = \sup_{\frac{1}{2}Q} |h| \geq e^{-N_0}$. Changing h into $-h$ we may assume that $h(x_0) > 0$. The point x_0 belongs to a certain q_0. Therefore $q_0 \cap \frac{1}{2}Q \neq \emptyset$. By the above estimate we have $\text{osc}_{q_0} h \leq \tau\big(\frac{3}{K}\big) \text{osc}_Q h$. Now,

$$\text{osc}_Q h = \sup_Q h - \inf_Q h \leq \sup_Q |h| + \sup_Q (-h) \leq 2 \sup_Q |h| \leq 2.$$

It follows that $\text{osc}_{q_0} h \leq 2\tau(\frac{3}{K})$. Then,

$$\inf_{q_0} h = \sup_{q_0} h - \text{osc}_{q_0} h \geq e^{-N_0} - 2\tau\big(\frac{3}{K}\big) \geq m_0 > 0$$

si $K >> 1$. We fix K.

Let us show (5.12). Let q be a cube of the partition. By (5.3) we have $Q \subset 2Kq \subset 3Q$. Taking K of the form $2^{\ell-1}$ we obtain, $Q \subset 2^\ell q$. Now from the hypothesis we have,

$$\frac{\sup_{2q} |h|}{\sup_q |h|} \leq N_0 \Longleftrightarrow \sup_q |h| \geq \frac{1}{N_0} \sup_{2q} |h|.$$

Iterating this inequality we get,

$$\sup_q |h| \geq \frac{1}{N_0^\ell} \sup_{2^\ell q} |h| \geq \frac{1}{N_0^\ell} \sup_Q |h| \geq \frac{\lambda}{N_0^\ell}.$$

□

Let us go back to the proof of Proposition 5.6.

We start from the cube Q which we cut into K^d equal cubes $Q = \cup_i q_i^{(1)}$. Lemma 5.7 with $\lambda = 1$ implies that there exists i_1 such that $\{x \in Q : |h(x)| < m_0\} \subset \cup_{i \neq i_1} q_i^{(1)}$. If we remove $q_{i_1}^{(1)}$ it remains $K^d - 1$ cubes. Each cube has measure $\frac{1}{K^d}|Q|$ so $|\{x \in Q : |h(x)| < m_0\}| \leq (1 - \frac{1}{K^d})|Q|$. Let us divide each small cube $q = q_i^{(1)}$ (with $i \neq i_1$) into K^d equal cubes. We may apply Lemma 5.7 with $\lambda = b$. So there exists $q_{i_2}^{(2)} \subset q$ such that $\inf_{q_{i_2}^{(2)}} h \geq m_0 b$. So

$\{x \in q : |h(x)| < m_0 b\} \subset \cup_{i \neq i_2} q_i^{(2)}$. Since $m_0 b < m_0$ the set $\{x \in Q : |h(x)| < m_0 b\}$ is contenained in a union of at most $(K^d - 1) \times (K^d - 1)$ cubes having each a measure $\big(\frac{1}{K^d}\big)^2$. Therefore,

$$|\{x \in Q : |h(x)| < m_0 b\}| \leq \big(1 - \frac{1}{K^d}\big)^2.$$

We pursue applying Lemma 5.7 with $\lambda = b^2, \dots, b^{\ell-1}$; we find that,

$$|\{x \in Q : |h(x)| < m_0 b^\ell\}| \leq \big(1 - \frac{1}{K^d}\big)^{\ell+1} |Q|.$$

Let $a > 0$ be so large that $\frac{1}{m_0} e^{-a} < 1$. There exists a unique $\ell \in \mathbf{N}$ such that $b^{\ell-1} \leq \frac{1}{m_0} e^{-a} \leq b^\ell$. By the above argument we have, $|\{x \in Q : |h(x)| < e^{-a}\}| \leq \big(1 - \frac{1}{K^d}\big)^{\ell+1} |Q|$. Now there exists $\gamma > 0$ such that $b^\gamma = 1 - \frac{1}{K^d}$. It follows that,

$$|\{x \in Q : |h(x)| < e^{-a}\} \leq b^{\gamma(\ell+1)} |Q| = b^2 b^{\gamma(\ell-1)} |Q| \leq b^2 \big(\frac{e^{-a}}{m_0}\big)^\gamma |Q| \leq \frac{b^2}{m_0^\gamma} e^{-\gamma a} |Q|.$$

5.3. **End of the proof of Theorem 5.5.** The induction argument consists in cutting the cubes into smallest cubes and to find a cube having a small doubling index.

We begin by some useful Lemmas.

5.3.1. *Distribution of the doubling indices.* Let Q_0 be the unit cube and $f \in C^0(\overline{Q})$. For every cube q such that $2q \subset Q_0$ we have set,

$$N_f(q) = \operatorname{Log} \frac{\sup_{2q} |f|}{\sup_q |f|}.$$

Lemma 5.8. *Let Qbe a cube, $Q \subset Q_0$. Assume that Q is cut into K^d equal cubes q_i where $K \geq 24$. Set $N_{min} = \min_i N_f(q_i)$. Then,*

$$N_f(\frac{1}{2}Q) \geq \frac{K}{8} N_{min}.$$

Proof. There exists $x_0 \in \frac{1}{2}Q$ such that $\sup_{\frac{1}{2}Q} |f| = f(x_0)$. We have $x_0 \in q_{i_0}$ for a certain i_0. Using Lemma 4.5 with $m = 0$ we find that $2q_{i_0} \subset \big(\frac{1}{2} + \frac{3}{K}\big)Q$.

Now since $N_f(q_{i_0}) \geq N_{min}$ and $x_0 \in q_{i_0}$ we have,

$$\sup_{2q_{i_0}} |f| \geq e^{N_{min}} \sup_{q_{i_0}} |f| \geq e^{N_{min}} |f(x_0)|.$$

There exists a point $x_1 \in 2q_{i_0}$ such that $|f(x_1)| = \sup_{2q_{i_0}} |f|$. So $|f(x_1)| \geq e^{N_{min}} |f(x_0)|$. This point x_1 belongs to another cube q_{i_1} and since $N_f(q_{i_1}) \geq N_{min}$ we have,

$$\sup_{2q_{i_1}} |f| \geq e^{N_{min}} \sup_{q_{i_1}} |f| \geq e^{N_{min}} |f(x_1)| \geq e^{2N_{min}} |f(x_0)|.$$

There exists a point $x_2 \in 2q_{i_1}$ such that $\sup_{2q_{i_1}} |f| = |f(x_2)|$. This point satisfies,

$$|f(x_2)| \geq e^{2N_{min}} |f(x_0)|.$$

By construction $x_1 \in q_{i_1} \cap 2q_{i_0} \subset q_{i_1} \cap \big(\frac{1}{2} + \frac{3}{K}\big)Q$. Using Lemma 4.5 with $m = 1$ we see that $2q_{i_1} \subset \big(\frac{1}{2} + \frac{3 \cdot 2}{K}\big)Q$. So $x_2 \in \big(\frac{1}{2} + \frac{3 \cdot 2}{K}\big)Q$. We construct a sequence (x_j) such that $|f(x_j)| \geq$

$e^{jN_{min}}|f(x_0)|$, $x_j \in \left(\frac{1}{2}+\frac{3j}{K}\right)Q$. We go until $j = \left[\frac{K}{6}\right]$. Since,

$$\frac{1}{2}+\frac{3}{K}\left[\frac{K}{6}\right] \leq \frac{1}{2}+\frac{3}{K}\frac{K}{6} = 1$$

the last point $\overline{x}$ belongs to Q and,

$$\sup_{Q}|f| \geq |f(\overline{x})| \geq e^{\left[\frac{K}{6}\right]N_{min}} \sup_{\frac{1}{2}Q}|f|.$$

Now, $\left[\frac{K}{6}\right] \geq \frac{K}{6} - 1 \geq \frac{K}{8}$ if $K \geq 24$. It follows that, $N_f(\frac{1}{2}Q) \geq \frac{K}{8}N_{min}$. □

Corollary 5.9. *Let $L = div(A\nabla)$ be uniformly elliptic in $2Q_0$ and let h be a bounded continuous solution of $Lh = 0$ in $2Q_0$. There exist constants N_0, J_0 such that if $Q \subset Q_0$ is cut into J^d equal cubes q_i with $J \geq J_0$ and $\mathcal{N}_h(Q) \geq N_0$ then for at least one cube q we have,*

$$\mathcal{N}_h(q) \leq \frac{1}{2}\mathcal{N}_h(Q).$$

Proof. We know that, $N_h(q) \leq \mathcal{N}_h(q) \leq a_1 N_h(q) + a_2$. By the previous Lemma there exists a cube q such that, $N_h(q) \leq \frac{8}{J}\mathcal{N}_h(\frac{1}{2}Q) \leq \frac{8}{J_0}\mathcal{N}_h(Q)$. Then,

$$\mathcal{N}_h(q) \leq \frac{8a_1}{J_0}\mathcal{N}_h(Q) + \frac{a_2}{N_0}\mathcal{N}_h(Q) \leq \frac{1}{2}\mathcal{N}_h(Q)$$

if J_0 and N_0 are large enough. □

5.3.2. *Notations.* We fix the ellipticity and Lipschitz constants Λ, C and we consider $Lh = \mathrm{div}(A\nabla)h = 0$ in $2Q_0$ where Q_0 is a cube with volume 1. We are going to vary the parameters $N \geq 1$ and $a > 0$ and our aim is to prove that,

$$E_a(h) = \{x \in Q_0 : |h(x)| < e^{-a}\sup_{Q_0}|h|\} \Longrightarrow |E_a(h)| \leq Ce^{-\frac{\beta a}{N}}|Q_0|. \tag{5.13}$$

We set,

$$m(u,a) = |\{x \in Q_0 : |u(x)| < e^{-a}\sup_{Q_0}|u|\}|, \quad M(N,a) = \sup m(u,a)$$

where the sup is taken on all operators $L = \mathrm{div}(A\nabla)$ and all u such that in Q_0,

(*i*) $A(x)$ is a uniformly elliptic symmetric matrix, with Lipschitz coefficients whose the ellipticity and Lipschitz constants are controlled by Λ and C,

(*ii*) u is a solution of $Lu = 0$ in $2Q_0$,

(*iii*) $\mathcal{N}_u(Q_0) \leq N$.

The goal is to prove that $M(N,a) \leq Ce^{-\frac{\beta a}{N}}$ where C and β will be independent of N.

Thanks to the Cases 1 and 2 considered before, we may assume that,

$$a >> 1, \quad \frac{a}{N} > c_0, \quad N > N_0 >> 1.$$

The first step consists in proving an induction relation on $M(N,a)$ and the second one that this relation implies (5.13).

5.3.3. *The induction relation.* We show that there exists $a_0 > 0$ and $0 < s < 1$ such that,

$$M(N,a) \leq M(\frac{1}{2}N, a - Na_0) + sM(N, a - Na_0). \tag{5.14}$$

Let u be a bounded continuous solution of $Lu = 0$ in $2Q_0$ with $\mathcal{N}_u(Q_0) \leq N$. Cut Q_0 into J^d equal cubes q where $J = 2^\ell$. If $\mathcal{N}_u(Q_0) \leq \frac{1}{2}N$ then for all $q \subset Q_0$ we have $\mathcal{N}_u(q) \leq \frac{1}{2}N$. If contrariwise $\mathcal{N}_u(Q_0) > \frac{1}{2}N \geq \frac{1}{2}N_0 >> 1$ we can apply Corollary 5.9 and deduce that there exists a sub cube q_0 such that $\mathcal{N}_u(q_0) \leq \frac{1}{2}N$. Since the union $Q = \cup q$ is disjoint (up to a set of measure zero) we have,

$$m(u,a) = \sum_q |\{x \in q : |u(x)| < e^{-a} \sup_{Q_0} |u|\}.$$

We shall show that there exists $a_0 > 0$ depending on J ($a_0 \approx \ell$) such that,

$$\sup_q |u| \geq e^{-a_0 N} \sup_{Q_0} |u|. \tag{5.15}$$

Indeed by Lemma 4.5 we have $Q_0 \subset 2Jq$ and $Jq \subset 2Q_0$, where $J = 2^\ell$. By definition we have,

$$\sup_q |u| \geq e^{-N} \sup_{2q} |u|.$$

Iterating this inequality we get,

$$\sup_q |u| \geq e^{-(\ell+1)N} \sup_{2^{\ell+1}q} |u| \geq e^{-(\ell+1)N} \sup_{Q_0} |u|.$$

Then,

$$\begin{aligned}
m(u,a) &\leq \sum_q |\{x \in q : |u(x)| \leq e^{-a+a_0N} \sup_q |u|\}|, \\
&\leq |\{x \in q_0 : |u(x)| \leq e^{-a+a_0N} \sup_{q_0} |u|\}| + \sum_{q \neq q_0} |\{x \in q : |u(x)| \leq e^{-a+a_0N} \sup_q |u|\}|, \\
&\leq (1) + (2).
\end{aligned}$$

Let us estimate the term (1). The problem is that $|q_0| = J^{-d} \neq 1$. Let $\widetilde{q_0} = \{y = Jx : x \in q_0\}$. Then,

$$|\widetilde{q_0}| = \int_{\widetilde{q_0}} dy = J^d \int_{q_0} dx = J^d |q_0| = 1.$$

Set $v(y) = u(x) = u\big(\frac{y}{J}\big)\big)$, $\quad y \in \widetilde{q_0}$. We have,

$$\operatorname{div}\big(A(\frac{y}{J})\nabla_y v\big)(y) = \frac{1}{J^2}\operatorname{div}\big(A(x)\nabla_x u\big)(x) = 0.$$

Now, since $J \geq 1$,

$$|A(\frac{y}{J}) - A(\frac{y'}{J})| \leq C\frac{|y-y'|}{J} \leq C|y-y'|,$$

Eventually, $\langle A\big(\frac{y}{J}\big)\xi, \xi\rangle \geq \Lambda|\xi|^2$ et $\mathcal{N}_v(\widetilde{q_0}) = \mathcal{N}_u(q_0) \leq \frac{1}{2}N$.

We have,

$$|\{y\in\widetilde{q_0}:|v(y)|\le e^{-a+a_0N}\sup_{\widetilde{q_0}}|v(y)|\}| = \int_{\{y\in\widetilde{q_0}:|v(y)|\le e^{-a+a_0N}\sup_{\widetilde{q_0}}|v|\}}dy,$$
$$= J^d\int_{\{x\in q_0:|u(x)|\le e^{-a+a_0N}\sup_{q_0}|u|\}}dx.$$

Since $|\widetilde{q_0}|=1$ the left hand side is bounded by $M(\frac{1}{2}N, a-a_0N)$ so,

$$(1)\le J^{-d}M(\frac{1}{2}N,a-a_0N).$$

We use the same argument for the term (2) except that here we have $\mathcal{N}_u(q)\le N$. We obtain, since J^d-1 terms are remaining in the sum,

$$(2)\le (J^d-1)J^{-d}M(N,a-a_0N) = (1-\frac{1}{J^d})M(N,a-a_0N) = sM(N,a-a_0N)$$

with $s<1$, which ends the proof of (5.14).

5.3.4. *The induction relation implies Theorem 5.5.* Our goal is to show now that,

$$M(N,a)\le Ce^{-\frac{\beta a}{N}} \tag{5.16}$$

where $C>0$ is large enough, $\beta>0$ small enough, by a double induction on N and a.

Recall that (5.16) is true in the two cases : (i) $N\le N_0\quad\forall a>0$, (ii) $\frac{a}{N}\le c_0$.

Without loss of generality we may assume that : $N=2^\ell, \ell\ge\ell_0$, $a=ka_02^\ell$. We show that,

$$\Big((5.16)\quad\text{true for } N=2^{\ell-1}\text{ for all } a\Big)\Longrightarrow\Big((5.16)\quad\text{true for } N=2^{\ell}\text{ for all } a\Big)$$

Since $\frac{a}{N}=ka_0$ the reminder (ii) shows that (5.16) is true if $k\le k_0:=\frac{c_0}{a_0}$.

We describe the induction step going from $(k-1)a_02^\ell$ to ka_02^ℓ. By the induction we have,

$$\begin{aligned} &M(2^\ell,(k-1)a_02^\ell)\le Ce^{-\beta(k-1)a_0},\\ &M(2^{\ell-1},(k-1)a_02^\ell)\le e^{-2\beta(k-1)a_0}.\end{aligned} \tag{5.17}$$

We apply (5.14) and (5.16) we get, with $s<1$,

$$\begin{aligned} M(2^\ell,ka_02^\ell)&\le M(2^{\ell-1},(k-1)a_02^\ell)+sM(2^\ell,(k-1)a_02^\ell),\\ &\le Ce^{-2\beta(k-1)a_0}+sCe^{-\beta(k-1)a_0}.\end{aligned}$$

The goal is to show that,

$$e^{-2\beta(k-1)a_0}+se^{-\beta(k-1)a_0}\le e^{-k\beta a_0}$$

for $k\ge k_0$ and a certain $\beta>0$. Dividing by $e^{-ka_0\beta}$ we are left with,

$$e^{-(k-2)\beta a_0}+se^{\beta a_0}\le 1.$$

We choose β such that $se^{\beta a_0}\le\frac{1+s}{2}$ that is, $e^{\beta a_0}\le\frac{1}{2}+\frac{1}{2s}$ or, $\beta a_0\le \mathrm{Log}(\frac{1}{2}+\frac{1}{2s})$ which is possible since $\frac{1}{2}+\frac{1}{2s}>1$, then we take k_0 so large that $e^{-(k-2)\beta a_0}\le\frac{1-s}{2}$.

□

6. Appendix

In what follows we prove Lemma 4.5 and we recall some properties of the solutions of second order elliptic equations in divergence form.

6.1. **Proof of Lemma 4.5.** Let us show (i). We may assume that,

$$Q=\{x\in\mathbf{R}^d:\left|x_j-\frac{L}{2}\right|\le\frac{L}{2}\}\quad q=\{x\in\mathbf{R}^d:\left|x_j-(i_j+\frac{1}{2})\frac{L}{K}\right|\le\frac{L}{2K},0\le i_j\le K-1\}.$$

We have first, $\left|\frac{L}{2}-(i_j+\frac{1}{2})\frac{L}{K}\right|\le\frac{L}{2}-\frac{L}{2K}$. Indeed,

$$\frac{L}{2}-(i_j+\frac{1}{2})\frac{L}{K}\le\frac{L}{2}-\frac{L}{2K},\quad \frac{L}{2}-(i_j+\frac{1}{2})\frac{L}{K}\ge\frac{L}{2}-(K-\frac{1}{2})\frac{L}{K}\ge-\frac{L}{2}+\frac{L}{2K}.$$

If $x\in Q$ we can write,

$$\left|x-(i_j+\frac{1}{2})\frac{L}{K}\right|\le\left|x-\frac{L}{2}\right|+\left|\frac{L}{2}-(i_j+\frac{1}{2})\frac{L}{K}\right|\le\frac{L}{2}+\frac{L}{2}-\frac{L}{2K}\le L=2K\frac{L}{2K}.$$

If $x\in 2Kq$ we can write,

$$\left|x-\frac{L}{2}\right|\le\left|x-(i_j+\frac{1}{2})\frac{L}{K}\right|+\left|\frac{L}{2}-(i_j+\frac{1}{2})\frac{L}{K}\right|\le L+\frac{L}{2}-\frac{L}{2K}\le 3\frac{L}{2}.$$

Eventually, if $x\in Kq$ we can write,

$$\left|x-\frac{L}{2}\right|\le\left|x-(i_j+\frac{1}{2})\frac{L}{K}\right|+\left|\frac{L}{2}-(i_j+\frac{1}{2})\frac{L}{K}\right|\le\frac{L}{2}+\frac{L}{2}-\frac{L}{2K}\le 2\frac{L}{2}.$$

Let us show (ii). We may assume that $Q=\{x\in\mathbf{R}^d:|x_j-\frac{1}{2}L|\le\frac{1}{2}L,\quad 1\le j\le d\}$. Then, $q=\{x:|x_j-(i_j+\frac{1}{2})\frac{L}{K}|\le\frac{L}{2K},1\le j\le d\}$.

Let $x\in q\cap(\frac{1}{2}+\frac{3m}{K})Q$. Assume that there exists j such that $i_j\frac{L}{K}>\frac{3m}{K}\frac{L}{2}+\frac{3L}{4}$. Then,

$$\begin{aligned}\left|x_j-\frac{L}{2}\right|&\ge\left|(i_j+\frac{1}{2})\frac{L}{K}-\frac{L}{2}\right|-\left|x_j-(i_j+\frac{1}{2})\frac{L}{K}\right|>\frac{3m}{K}\frac{L}{2}+\frac{3L}{4}+\frac{L}{2K}-\frac{L}{2}-\frac{L}{2K},\\&>(\frac{3m}{K}+\frac{1}{2})\frac{L}{2},\end{aligned}$$

so $x\notin(\frac{3m}{K}+\frac{1}{2})Q$, which is absurd; therefore for all j we have $i_j\le\frac{3K}{4}+\frac{3m}{2}$.

Likewise assume that there exists j such that $i_j+\frac{1}{2}<\frac{K}{4}-\frac{1}{2}-\frac{3m}{2}$. Then,

$$\begin{aligned}x_j-\frac{L}{2}&=x_j-(i_j+\frac{1}{2})\frac{L}{K}+(i_j+\frac{1}{2})\frac{L}{K}-\frac{L}{2}\le\frac{L}{2K}+\frac{L}{4}-\frac{L}{2K}-\frac{3mL}{2K}-\frac{L}{2}\\&\le-\frac{3mL}{2K}-\frac{L}{4}=-(\frac{3m}{K}+\frac{1}{2})\frac{L}{2},\end{aligned}$$

which is absurd. Therefore for all j we have $i_j\ge\frac{K}{4}-1-\frac{3m}{2}$. Summing up we must have,

$$\frac{K}{4}-1-\frac{3m}{2}\le i_j\le\frac{3K}{4}+\frac{3m}{2}. \tag{6.1}$$

We deduce hat,

$$-\frac{L}{4}-\frac{L}{2K}-\frac{3mL}{2K}\le(i_j+\frac{1}{2})\frac{L}{K}-\frac{L}{2}\le\frac{L}{4}+\frac{L}{2K}+\frac{3mL}{2K}.$$

Let $x\in 2q$. We have,

$$\begin{aligned}\left|x_j-\frac{L}{2}\right|&\le\left|x_j-(i_j+\frac{1}{2})\frac{L}{K}+(i_j+\frac{1}{2})\frac{L}{K}-\frac{L}{2}\right|,\\&\le\frac{L}{K}+\frac{L}{4}+\frac{L}{2K}+\frac{3mL}{2K}=(\frac{3(m+1)}{K}+\frac{1}{2})\frac{L}{2},\end{aligned}$$

that is, $2q \subset \big(\frac{3(m+1)}{K} + \frac{1}{2}\big)Q$.

Let us show (iii). We have, from (6.1), $\frac{K}{4} - 1 \leq i_j \leq \frac{3K}{4}$. If $x \in \frac{1}{3}Kq$ we have $|x_j - (i_j + \frac{1}{2})\frac{L}{K}| \leq \frac{1}{3}K\frac{L}{2K} = \frac{L}{6}$ so,

$$(i_j + \frac{1}{2})\frac{L}{K} - \frac{L}{6} \leq x_j \leq (i_j + \frac{1}{2})\frac{L}{K} + \frac{L}{6}.$$

It follows that,

$$(\frac{K}{4} - \frac{1}{2})\frac{L}{K} - \frac{L}{6} \leq x_j \leq (\frac{3K}{4} + \frac{1}{2})\frac{L}{K} + \frac{L}{6} \Longleftrightarrow \frac{1}{12}L - \frac{L}{2K} \leq x_j \leq \frac{11}{12}L + \frac{L}{2K}.$$

Therefore if K is large enough we have $0 < x_j < L$ so $x \in Q$.

6.2. **Some properties of the solutions of elliptic equations.** We consider in an open set Ω in $\mathbf{R}^d$ a symmetric matrix $A(x) = (a_{ij}(x))_{1 \leq i,j \leq d}$ with $L^\infty(\Omega)$ coefficients such that,

$$|\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j| \leq \Lambda|\xi|^2, \qquad \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, \forall \xi \in \mathbf{R}^d. \tag{6.2}$$

We shall denote in what follows, $L = \sum_{i,j=1}^d \partial_j\big(a_{ij}(x)\partial_i\big) = \mathrm{div}(A\nabla)$.

6.2.1. *Weak solution, sub-solution, super-solution.* A weak solution (resp. weak sub-solution faible, resp. weak super-solution) of L is an element $u \in H^1_{\mathrm{loc}}(\Omega)$ such that,

$$\sum_{i,j=1}^d \int_\Omega a_{ij}(x)\partial_i u(x)\partial_j\varphi(x)\,dx = 0 \quad (\textit{resp.} \leq 0, \mathrm{resp} \geq 0), \quad \forall \varphi \in H^1_0(\Omega), \quad \varphi \geq 0 \text{in } \Omega.$$

Remark 6.1. (i) in the définition above it is equivalent to take φ in $C^\infty_0(\Omega)$.

(ii) For smooth functions this definition is equivalent to the fact that $Lu = 0$ (resp. $\geq 0, \leq 0$) in Ω.

Lemma 6.2. (i) *Let $\Phi \in W^{1,\infty}_{loc}(\mathbf{R})$ be a non increasing convex function. Let $u \in H^1_{loc}(\Omega)$ be a real valued* **weak solution** *of L. Let $v = \Phi(u)$. If $v \in H^1_{loc}(\Omega)$ then v is a weak sub-solution of L.*

(ii) *Let $\Phi \in W^{1,\infty}_{loc}(\mathbf{R})$ be a non decreasing and convex function. Let $u \in H^1_{loc}(\Omega)$ be a real valued* **weak sub-solution** *of L. Let $v = \Phi(u)$. If $v \in H^1_{loc}(\Omega)$ then v is a weak sub-solution of L.*

Proof. (i) Assume first that $\Phi \in C^2_{\mathrm{loc}}(\mathbf{R})$. The hypotheses imply that $\Phi'(s) \leq 0$ and $\Phi''(s) \geq 0$ for all $s \in \mathbf{R}$. Let $\varphi \in C^\infty_0(\Omega)$. We have,

$$\begin{aligned}\sum_{i,j=1}^d \int_\Omega a_{ij}\,\partial_i v\,\partial_j\,\varphi\,dx &= \sum_{i,j=1}^d \int_\Omega a_{ij}\,\Phi'(u)\partial_i u\,\partial_j\varphi\,dx = -\sum_{i,j=1}^d \int_\Omega a_{ij}\,\partial_i u\,\partial_j\big(-\Phi'(u)\varphi\big)\,dx \\ &\quad - \sum_{i,j=1}^d \int_\Omega \varphi\,\Phi''(u)\,a_{ij}\,\partial_i u\,\partial_j u\;dx = -(1) - (2).\end{aligned}$$

The function $\psi = -\Phi'(u)\varphi$ is non negative and belongs to $H^1_0(\Omega)$. Since u a solution the term (1) vanishes. The term (2) is non negative by (6.2) and the fact that $\Phi'' \geq 0$. Therefore the left hand side is non positive.

If $\Phi \in W^{1,\infty}_{\rm loc}(\mathbf{R})$ let $\Phi_\varepsilon = \rho_\varepsilon \star \Phi$ where ρ_ε an approximation of the identity. Then Φ_ε is C^2 and $\Phi'_\varepsilon = \rho_\varepsilon \star \Phi' \geq 0$. Moreover Φ_ε is convex. Indeed let $\lambda \in (0,1)$. Since Φ is convex and $\rho_\varepsilon \geq 0$ we have,

$$\begin{aligned}\Phi_\varepsilon(\lambda s_1 + (1-\lambda)s_2) &= \int \rho_\varepsilon(y)\Phi((\lambda(s_1 - y) + (1-\lambda)(s_2 - y))\, dy, \\ &\leq \lambda \int \rho_\varepsilon(y)\Phi((s_1 - y)\, dy + (1-\lambda)\int \rho_\varepsilon(y)\Phi(s_2 - y)\, dy, \\ &\leq \lambda\Phi_\varepsilon(s_1) + (1-\lambda)\Phi_\varepsilon(s_2).\end{aligned}$$

Therefore we can apply the result obtained in the first part that is,

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij}\, \partial_i \Phi_\varepsilon(u)\, \partial_j\, \varphi\, dx = \sum_{i,j=1}^{d} \int_\Omega a_{ij}\, \rho_\varepsilon \star \partial_i \Phi(u)\, \partial_j\, \varphi\, dx \leq 0. \tag{6.3}$$

By hypothesis $\partial_i \Phi(u) \in L^2_{\rm loc}(\Omega)$. Therefore $\rho_\varepsilon \star \partial_i \Phi(u)$ converges to $\partial_i \Phi(u)$ in $L^2_{\rm loc}(\Omega)$. Since $\varphi \in C_0^\infty(\Omega)$ we can pass to the limit in (6.3) and deduce that,

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij}\, \partial_i \Phi(u)\, \partial_j\, \varphi\, dx \leq 0.$$

(ii) The proof is the same. We have just to notice that $\sum_{i,j=1}^{d} \int_\Omega a_{ij}\, \partial_i u\, \partial_j \big(\Phi'(u)\varphi\big)\, dx \leq 0$ since u is a weak sub-solution. □

Remark 6.3. We have a similar result if $\Phi \in W^{1,\infty}_{\rm loc}((0,+\infty))$ and $u > 0$. The proof is the same.

Example 6.4. Consider the function defined on $(0,+\infty)$ by $\Phi(s) = (\mathrm{Log}\, s)^-$ that is $\Phi(s) = 0$ if $s \geq 1$, $\Phi(s) = -\mathrm{Log}\, s$ if $0 < s \leq 1$. This is a continuous function on $(0,+\infty)$, C^∞ on $(0,1) \cup (1+\infty)$, locally bounded, decreasing and convex. We have $\Phi'(s) = 0$ for $s > 1$ and $\Phi'(s) = -\frac{1}{s}$ for $0 < s < 1$.

6.2.2. *The Cacciopoli inequality.*

Lemma 6.5. *Let $u \in H^1_{loc}(\Omega)$ be a positive weak sub-solution of L and $\omega \subset\subset \Omega$ an open set. There exists $C > 0$ depending only on $\Omega, \omega, d, \Lambda, \lambda$ such that,*

$$\int_\omega |\nabla u(x)|^2\, dx \leq C \int_\Omega |u(x)|^2\, dx.$$

Proof. Let $\psi \in C_0^\infty(\Omega)$ be positive such that $\psi = 1$ on ω. The function $\varphi = \psi^2 u$ belongs to $H^1_0(\Omega)$ and it is positive. We have, by the definition of a sub-solution,

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij}\, \partial_i u\, \partial_j(\psi^2 u)\, dx \leq 0,$$

which implies that,

$$(1) = \sum_{i,j=1}^{d} \int_\Omega a_{ij}\, \psi^2\, \partial_i u\, \partial_j u\, dx \leq -2 \sum_{i,j=1}^{d} \int_\Omega \psi\, u\, a_{ij}\, \partial_i u\, \partial_j \psi\, dx = (2).$$

We have,

$$(1) \geq \lambda \int_\Omega \psi^2 |\nabla u|^2 \, dx.$$

Next,

$$|\sum_{i,j=1}^d a_{ij}\, \partial_i u\, \partial_j \psi| = |\langle A\nabla u, \nabla\psi\rangle| \leq \Lambda |\nabla u||\nabla \psi|,$$

so that using the Cauchy-Schwarz inequality we obtain,

$$|(2)| \leq 2\Lambda \Big(\int_\Omega \psi^2 |\nabla u|^2 \, dx \Big)^{\frac{1}{2}} \Big(\int_\Omega u^2 |\nabla \psi|^2 \, dx \Big)^{\frac{1}{2}}.$$

Using these estimates and the fact that $\psi = 1$ on ω we deduce the lemma. □

Remark 6.6. If $\Omega = B(x_0, R)$ and $\omega = B(x_0, r)$ with $r < R$ then $C = \frac{C'}{(R-r)^2}$ where C' depends only on d, Λ, λ.

6.2.3. *Moser iteration.* We denote in what follows $B(x_0, r)$ the ball centered at x_0 with radius $r > 0$.

Theorem 6.7. *Let $x_0 \in \Omega$ and $0 < r < \rho$ be such that $B(x_0, \rho) \subset \Omega$. There exists $C > 0$ such that for all positive sub-solution $u \in H^1_{loc}(\Omega)$ of L we have,*

$$\|u\|_{L^\infty(B(x_0,r))} \leq C \|u\|_{L^2(B(x_0,\rho))}. \tag{6.4}$$

Corollary 6.8. *Let $x_0 \in \Omega, r > 0$ such that $B(x_0, 3r) \subset \Omega$. Then there exists $C \geq 1$ depending only on d, λ, Λ such that for all positive sub-solution v of L in Ω we have,*

$$C^{-1} r^{-\frac{d}{2}} \|v\|_{L^2(B(x_0,r)} \leq \|v\|_{L^\infty(B(x_0,r)} \leq C r^{-\frac{d}{2}} \|v\|_{L^2(B(x_0,2r)}. \tag{6.5}$$

Proof of the Corollary. We apply the inequality (6.4) with $x_0 = 0, \Omega = B(0,3), r = 1, \rho = 2$ to the function $u(y) = v(x_0 + ry)$. Then u is a solution of another elliptic equation having the same constants λ, Λ. Moreover we have,

$$\|v\|_{L^2(B(x_0,r))} = r^{\frac{d}{2}} \|u\|_{L^2(B(0,1))} \quad \text{et} \quad \|v\|_{L^\infty(B(x_0,r))} = \|u\|_{L^2(B(0,1))}.$$

□

Proof of Theorem 6.7 . Consider a sequence of balls $B_j = B(x_0, r_j)$ with $r_j = r + (\rho - r)2^{-j}$, so that,

$$B_{j+1} \subset B_j \subset \cdots \subset B_0 = B(x_0, \rho) \quad \text{and} B_\infty = \cap_{j\in\mathbf{N}} B_j = \overline{B(x_0, r)}.$$

The method of proof consists in proving that there exists $\kappa > 1$ such that we can estimate $\|u\|_{L^{2\kappa^{j+1}}(B_{j+1})}$ by $\|u\|_{L^{2\kappa^j}(B_j)}$. The existence of κ comes from the following corollary of the Sobolev embedding.

Lemma 6.9. *Let $\kappa \in [1, \frac{d}{d-2}]$ for $d \geq 2$, $\kappa \in [1, +\infty)$ for $d = 2$. There exists $C > 0$ such that for any ball B and any positive $v \in H^1(B)$ we have,*

$$\|v^\kappa\|^2_{L^2(B)} \leq C \big(\|\nabla v\|^{2\kappa}_{L^2(B)} + \|v\|^{2\kappa}_{L^2(B)} \big).$$

Proof. The Sobolev inequality implies that,

$$\|v\|^{2\kappa}_{L^{2\kappa}(B)} = \|v^\kappa\|^2_{L^2(B)} \leq C \|v\|^{2\kappa}_{H^1(B)} \leq C \big(\|\nabla v\|_{L^2(B)} + \|v\|_{L^2(B)} \big)^{2\kappa}.$$

We have just to use the inequality $(a+b)^{2\kappa} \leq 2^{2\kappa}(a^{2\kappa} + b^{2\kappa})$. □

Lemma 6.10. *Let $\kappa \in (1, \frac{d}{d-2}]$ for $d \geq 2$, $\kappa \in (1, +\infty)$ for $d = 2$. Assume that $u \in H^1(B_j)$ Let u be a weak positive sub-solution of L. Then u^κ belongs to $H^1(B_{j+1})$ and it is a weak positive sub-solution of L in B_{j+1}. Moreover,*

$$\|u^\kappa\|^2_{L^2(B_{j+1})} \leq C(2^{2j\kappa} + 1)\|u\|^{2\kappa}_{L^2(B_j)} \tag{6.6}$$

where $C > 0$ depends only $d, \lambda, \Lambda, r, \rho$.

Proof. **Step 1.** Let $\Phi : [0, +\infty) \to [0, +\infty)$ be defined by $\Phi(s) = s^\kappa$. From the previous lemma we have $\Phi(u) \in L^2(B_j)$.

For $n \in \mathbf{N}^*$ and $s \geq 0$ set $\theta_n(s) = (s + \frac{1}{n})^\kappa$ and,

$$\Phi_n(s) = \begin{cases} \theta_n(s), & 0 \leq s \leq n, \\ \theta_n'(n)s + \theta_n(n) - n\theta_n'(n), & s > n. \end{cases}$$

Notice that since $\kappa > 1$ we have $\theta_n''(s) > 0$ for all $s \geq 0$. The Taylor formula implies that,

$$\theta_n(s) = \theta_n(n) + (s-n)\theta_n'(n) + (s-n)^2 \int_0^1 (1-\lambda)\, \theta_n''(\lambda s + (1-\lambda)n)\, d\lambda,$$

so that,

$$\theta_n(n) + (s-n)\theta_n'(n) \leq \theta_n(s), \quad s \geq 0.$$

We deduce that for all $s \geq 0$ we have,

$$\Phi_n(s) \leq \theta_n(s) \leq 2^\kappa(s^\kappa + \frac{1}{n^\kappa}) \leq 2^\kappa(\Phi(s) + 1). \tag{6.7}$$

Step 2. The function Φ_n is C^1, non decreasing, $\Phi_n' \in L^\infty(0, +\infty)$ $\Phi_n'' \in L^\infty(0, +\infty)$ and Φ_n is convex. Indeed we have,

$$\Phi_n'(s) = \begin{cases} \theta_n'(s), & 0 \leq s \leq n, \\ \theta_n'(n), & s > n, \end{cases} \qquad \Phi_n''(s) = \begin{cases} \theta_n''(s), & 0 \leq s \leq n, \\ 0, & s > n. \end{cases}$$

Step 3. $\Phi_n(u) \in H^1(B_j)$. First by Step 1 and (6.7) we have $\Phi_n(u) \in L^2(B_j)$. Next, $\nabla\Phi_n(u) = \Phi_n'(u)\nabla u \in L^2(B_j)$ since $u \in H^1(B_j)$ and $\Phi_n'(u) \in L^\infty(B_j)$.

Step 4. $\Phi_n(u)$ is a weak sub-solution of L. This results from the previous steps and from Lemma 6.2.

Step 5. The sequence $(\Phi_n(u))$ converges to $\Phi(u)$ in $L^2(B_j)$.

Indeed, for all $s_0 \geq 0$ $(\Phi_n(s_0))$ converges to $\Phi(s_0)$. Then, from (6.7) we have, $\Phi_n(u) \leq 2^\kappa(\Phi(u) + 1)$. Therefore,

$$|\Phi_n(u) - \Phi(u)|^2 \leq C'(\Phi(u)^2 + 1) \in L^1(B_j).$$

We apply the dominated convergence theorem to conclude.

Step 6. $\Phi(u) \in H^1(B_{j+1})$. Indeed, first by Step 1. we have $\Phi(u) \in L^2(B_{j+1})$. Next, Step4., Lemma 6.5 and (6.7) imply that,

$$\|\nabla\Phi_n(u)\|_{L^2(B_{j+1})} \leq C\|\Phi_n(u)\|_{L^2(B_j)} \leq C'\|1 + \Phi(u)\|_{L^2(B_j)}. \tag{6.8}$$

The sequence $(\nabla\Phi_n(u))$ is therefore uniformly bounded in $L^2(B_{j+1})$. Then there exists a subsequence such that $(\nabla\Phi_{\sigma(n)}(u)))$ converges weakly to $v \in L^2(B_{j+1})$. On the other hand, by Step 5. $(\nabla\Phi_{\sigma(n)}(u)))$ converges to $\nabla\Phi(u)$ in $\mathcal{D}'(B_{j+1})$. We deduce that $\nabla\Phi(u) = v \in L^2(B_{j+1})$, so $\Phi(u) \in H^1(B_{j+1})$.

Step 7. Since Φ is non decreasing, convex, and $\Phi(u) \in H^1(B_{j+1})$, Lemma 6.2 shows that $\Phi(u)$ is a weak sub-solution of L. Since the difference between the radius of B_j and that of B_{j+1} is proportional to 2^{-j} it follows from Lemma 6.5 and from Remark 6.6 that,

$$\int_{B_{j+1}} |\nabla \Phi(u)|^2 \, dx \leq K 2^{2j} \int_{B_j} |\Phi(u)|^2 \, dx.$$

Using Lemma 6.9 with $v = u$ we deduce the inequality (6.6). □

We introduce then the sequence of functions defined by,

$$w_j = u^{\kappa^j}.$$

If $u \in H^1(B_0)$ is a positive weak sub-solution of L we obtain, by induction, that w_j belongs to $H^1(B_j)$ and it is a weak sub-solution of L in B_j. Notice that $w_{j+1} = (w_j)^\kappa$. Set,

$$N_j = \left(\|w_j\|_{L^2(B_j)}\right)^{\frac{1}{\kappa^j}}.$$

Using (6.6) we get,

$$N_{j+1}^{2\kappa^{j+1}} = \|w_{j+1}\|^2_{L^2(B_{j+1})} = \|w_j^\kappa\|^2_{L^2(B_{j+1})} \leq C(2^{2j\kappa}+1)\|w_j\|^{2\kappa}_{L^2(B_j)} = C(2^{2j\kappa}+1) N_j^{2\kappa^{j+1}}.$$

Therefore,

$$N_{j+1}^2 \leq \Big(C(2^{2j\kappa}+1)\Big)^{\frac{1}{\kappa^{j+1}}} N_j^2. \tag{6.9}$$

We have,

$$\prod_{j=0}^{J} C^{\frac{1}{\kappa^{j+1}}} = C^{\sum_{j=0}^{J} \left(\frac{1}{\kappa}\right)^{j+1}} \leq C^{\frac{1}{\kappa-1}}. \tag{6.10}$$

On the other hand set, $A_J = \prod_{j=0}^{J}(2^{2j\kappa}+1)^{\frac{1}{\kappa^{j+1}}}$. Since $1 + 2^{j\kappa} \leq 2^{j\kappa+1}$ we have,

$$\operatorname{Log} A_J = \sum_{j=0}^{J} \frac{1}{\kappa^{j+1}} \operatorname{Log}(1+2^{j\kappa}) \leq \operatorname{Log} 2 \sum_{j=0}^{+\infty} \frac{j\kappa+1}{\kappa^{j+1}} = c_0.$$

We deduce that $A_J \leq e^{c_0}$. Using this inequality together with (6.9), (6.10) we obtain,

$$\limsup_{j\to+\infty} N_j^2 \leq C^{\frac{1}{\kappa-1}} e^{c_0} N_0^2.$$

This shows that the sequence (N_j) is bounded. We shall deduce that u belongs to $L^\infty(B(x_0,r))$. Indeed set $M = \sup N_j$. Then by définition of w_j and N_j we have,

$$\int_{B(x_0,r)} |u|^{2\kappa^j} \, dx \leq \int_{B_j} |u|^{2\kappa^j} \, dx \leq M^{2\kappa^j}.$$

Set,

$$A = \{x \in B(x_0,r) : |u(x)| > 2M\}.$$

Then,

$$|A|(2M)^{2\kappa^j} \leq \int_{B(x_0,r)} |u|^{2\kappa^j} \, dx.$$

Combining these two inegalities we deduce that $|A| \leq 2^{-2\kappa^j}$ for all $j \in \mathbf{N}$ which implies that $|A| = 0$. This shows that M is an essential supremum of u. Therefore $u \in L^\infty(B(x_0,r))$. Moreover M is bounded by a multiple of N_0 which is the L^2 norm of u on the ball $B(x_0,\rho)$. □

Remark 6.11. Let Q be a cube. There exists $C > 0$ such that for all positive weak sub-solution $u \in H^1_{\rm loc}(\Omega)$ of L in $2Q$ we have,

$$\sup_{\frac{1}{2}Q} u \le C\|u\|_{L^2(Q)}. \tag{6.11}$$

The proof is similar to that in the case of balls. We have just to work with the cubes $Q_j = \frac{1}{2}(1+2^{-j})Q$. We have $Q_{j+1} \subset Q_j, Q_0 = Q, Q_\infty = \frac{1}{2}Q$.

6.2.4. *A result about the oscillations.* We recall that the oscillation of a bounded function on a set Ω is defined by,

$$\mathrm{osc}_\Omega u = \sup_\Omega u - \inf_\Omega u.$$

Notice that if $\Omega_1 \subset \Omega_2$ we have,

$$\mathrm{osc}_{\Omega_1} u \le \mathrm{osc}_{\Omega_2} u.$$

Theorem 6.12. *Let Q be a cube and u be a bounded continuous solution of $Lu = 0$ in $2Q$. Then there exists $\gamma = \gamma(d, \Lambda) \in (0, 1)$ such that,*

$$osc_{\frac{1}{2}Q} u \le \gamma\, osc_Q u.$$

Corollary 6.13. *Let h be a bounded continuous solution of $Lh = 0$. There exists for small $s > 0$ a positive function $\tau(s)$ depending only on d, A such that $\tau(s) \to 0$ when $s \to 0$ and for all $Q \subset \Omega$,*

$$osc_{sQ} h \le \tau(s) osc_Q h \tag{6.12}$$

Proof of Theorem 6.12. The proof needs several steps.

Step1. Recall that there exists $C > 0$ such that for all positive weak sub-solution $v \in H^1(Q)$ of L we have,

$$\sup_{\frac{1}{2}Q} v \le C\|v\|_{L^2(Q)}. \tag{6.13}$$

Step 2. For all $\varepsilon > 0$ there exists $C = C(\varepsilon, d)$ such that for all $u \in H^1(Q)$ such that $|\{x \in Q : u = 0\}| \ge \varepsilon|Q|$ we have,

$$\int_Q |u|^2\,dx \le C \int_Q |\nabla u|^2\,dx.$$

Indeed, otherwise there exists $\varepsilon_0 > 0$ and a sequence $(u_k)_{k\in\mathbf{N}}$ such that:

$$|\{x \in Q : u_k = 0\}| \ge \varepsilon_0|Q|, \quad \int_Q |u_k|^2\,dx = 1, \quad \int_Q |\nabla u_k|^2\,dx \to 0.$$

Therefore $(u_k)_k$ is a bounded sequence in $H^1(Q)$. Then there exists a sub-sequence $(u_{\sigma(k)})_k$ which converges weakly to u_0 in $H^1(Q)$, so by compactness, it converges strongly in $L^2(Q)$. We have $\int_Q |u_0|^2\,dx = 1$. On the other hand, in $\mathcal{D}'(Q)$ the sequence $(\nabla u_{\sigma(k)})_k$ converges to ∇u_0 and to zero. So u_0 is non vanishing constant. Then,

$$\int_Q |u_{\sigma(k)} - u_0|^2\,dx \ge \int_{\{u_{\sigma(k)}=0\}} |u_{\sigma(k)} - u_0|^2\,dx \ge |u_0|^2\varepsilon_0|Q|.$$

The left hand side converges to zero while the right hand side is strictly positive, which is a contradiction.

Step 3. Let $u \in H^1(2Q)$ be a positive solution such that, $|\{x \in Q : u \geq 1\}| \geq \varepsilon |Q|$. Then there exists $C = C(\varepsilon, d, A) > 0$ such that $\inf_{\frac{1}{2}Q} u \geq C$.

Let $\rho > 0$ and $u_\rho = u + \rho$. Then, $u_\rho \geq \rho > 0$ and $\{x \in Q : u(x) \geq 1\} \subset \{x \in Q : u_\rho(x) \geq 1\}$ so,

$$|\{x \in Q : u_\rho(x) \geq 1\}| \geq |\{x \in Q : u(x) \geq 1\}| \geq \varepsilon |Q|$$

Set $v_\rho = (\operatorname{Log} u_\rho)^-$. We have, $v_\rho(x) = \begin{cases} 0, & \text{si } u_\rho(x) \geq 1, \\ \operatorname{Log} \frac{1}{u_\rho(x)} & \text{si } u_\rho(x) \leq 1. \end{cases}$

Since $u_\rho \geq \rho, v_\rho$ is non zero if and only if $\rho \leq u_\rho \leq 1$ so that,

$$0 \leq v_\rho(x) \leq \operatorname{Log} \frac{1}{\rho}, \quad \forall x \in Q.$$

It follows that $v_\rho \in L^2(Q)$. Let us show that $v_\rho \in H^1(Q)$. Since the function v_ρ is continuous there is no jump in the derivative. Therefore, $\partial_j v_\rho(x) = \begin{cases} 0, & \text{si } u_\rho(x) \geq 1, \\ -\frac{\partial_j u_\rho}{u_\rho} & \text{si } u_\rho(x) \leq 1. \end{cases}$ Since $u_\rho \geq \rho$ we deduce that,

$$|\partial_j v_\rho| \leq \left| \frac{\partial_j u_\rho}{u_\rho} \right| \leq \frac{1}{\rho} |\partial_j u_\rho| \in L^2(Q).$$

On the other hand, since v_ρ is a positive sub-solution, (6.11) implies that,

$$\sup_{\frac{1}{2}Q} v_\rho \leq C\Big(\int_Q v_\rho^2(x)\, dx \Big)^{\frac{1}{2}}.$$

Now, $|\{x \in Q : v_\rho(x) = 0\}| = |\{x \in Q : u_\rho(x) \geq 1\}| \geq \varepsilon |Q|$. Then Step 2 implies that there exists $C > 0$ such that,

$$\sup_{\frac{1}{2}Q} v_\rho \leq C\Big(\int_Q |\nabla v_\rho(x)|^2\, dx \Big)^{\frac{1}{2}}. \tag{6.14}$$

We are going to show that the right hand side is bounded. Let $\theta \in C_0^\infty(2Q)$ and $\theta = 1$ on Q. Take as a test function $\varphi = \frac{\theta^2}{u_\rho} \in H^1(Q)$. Then we have, skipping the summations,

$$0 = \int_{2Q} a_{ij} (\partial_i u_\rho) \partial_j \big(\frac{\theta^2}{u_\rho}\big)\, dx = -\int_{2Q} \theta^2 \frac{a_{ij}(\partial_i u_\rho)(\partial_j u_\rho)}{u_\rho^2}\, dx + 2 \int_{2Q} \frac{\theta a_{ij}(\partial_i u_\rho)(\partial_j \theta)}{u_\rho}\, dx.$$

We have,

$$\int_{2Q} \theta^2 \frac{a_{ij}(\partial_i u_\rho)(\partial_j u_\rho)}{u_\rho^2}\, dx \geq \lambda \int_{2Q} \theta^2 \left| \frac{\nabla u_\rho}{u_\rho} \right|^2 dx,$$

$$\left| \int_{2Q} \frac{\theta a_{ij}(\partial_i u_\rho)(\partial_j \theta)}{u_\rho}\, dx \right| \leq \Lambda \Big(\int_{2Q} \theta^2 \left| \frac{\nabla u_\rho}{u_\rho} \right|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{2Q} |\nabla \theta|^2\, dx \Big)^{\frac{1}{2}},$$

from which we deduce, since $\theta = 1$ on Q,

$$\int_Q \left| \frac{\nabla u_\rho}{u_\rho} \right|^2 dx \leq C \int_{2Q} |\nabla \theta|^2\, dx.$$

It follows that,

$$\int_Q |\nabla v_\rho|^2\, dx \leq \int_Q \left| \frac{\nabla u_\rho}{u_\rho} \right|^2 dx \leq C \int_{2Q} |\nabla \theta|^2\, dx.$$

We deduce from (6.14), since $v_\rho = (\operatorname{Log} u_\rho)^-$ that,

$$\sup_{\frac{1}{2}Q} v_\rho = \sup_{\frac{1}{2}Q} (\operatorname{Log} u_\rho)^- \leq C.$$

Then on $\frac{1}{2}Q$, either $u_\rho \geq 1$ or $u_\rho \leq 1$ and $-\operatorname{Log} u_\rho \leq C$, that means, $u_\rho \geq e^{-C}$ where C is independent of ρ. Letting ρ go to zero we obtain, $\inf_{\frac{1}{2}Q} u \geq C' > 0$.

Step4. End of the proof. Set,

$$\alpha_1 = \sup_Q u, \quad \beta_1 = \inf_Q u, \quad \alpha_2 = \sup_{\frac{1}{2}Q} u, \quad \beta_2 = \inf_{\frac{1}{2}Q} u.$$

Consider the positive solutions ,

$$\frac{u - \beta_1}{\alpha_1 - \beta_1}, \quad \text{ou} \quad \frac{\alpha_1 - u}{\alpha_1 - \beta_1}.$$

We have the following equalities,

$$A_1 := \{x \in Q : u(x) \geq \frac{1}{2}(\alpha_1 + \beta_1)\} = \{x \in Q : \frac{u(x) - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{2}\}$$
$$A_2 := \{x \in Q : u(x) < \frac{1}{2}(\alpha_1 + \beta_1) = \{x \in Q : \frac{\alpha_1 - u(x)}{\alpha_1 - \beta_1} > \frac{1}{2}\}.$$

Since $Q = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$ we have $|Q| = |A_1| + |A_2|$ so either $|A_1| \geq \frac{1}{2}|Q|$ or $|A_2| \geq \frac{1}{2}|Q|$.

Case 1. Assume that,

$$|A_1| = \left|\left\{x \in Q : \frac{2(u(x) - \beta_1)}{\alpha_1 - \beta_1} \geq 1\right\}\right| \geq \frac{1}{2}|Q|.$$

We apply Step 3. to the positive solution $\frac{2(u-\beta_1)}{\alpha_1-\beta_1}$. Then there exists $C > 1$ such that,

$$\inf_{\frac{1}{2}Q} \frac{u - \beta_1}{\alpha_1 - \beta_1} \geq \frac{1}{C}$$

from which we deduce that,

$$\beta_2 = \inf_{\frac{1}{2}Q} u \geq \beta_1 + \frac{1}{C}(\alpha_1 - \beta_1).$$

Case 2. Assume that,

$$|A_2| = \left|\left\{x \in Q : \frac{2(\alpha_1 - u(x))}{\alpha_1 - \beta_1} \geq 1\right\}\right| \geq \frac{1}{2}|Q|.$$

By the same argument we obtain,

$$\alpha_2 = \sup_{\frac{1}{2}Q} u \leq \alpha_1 - \frac{1}{C}(\alpha_1 - \beta_1).$$

Since $\beta_2 \geq \beta_1$ and $\alpha_2 \leq \alpha_1$ we get,

$$\text{in Case 1.} \quad \alpha_2 - \beta_2 \leq \alpha_1 - \big(\beta_1 + \frac{1}{C}(\alpha_1 - \beta_1)\big) = \big(1 - \frac{1}{C}\big)(\alpha_1 - \beta_1),$$
$$\text{in Case 2.} \quad \alpha_2 - \beta_2 \leq \alpha_2 - \beta_1 \leq \big(1 - \frac{1}{C}\big)(\alpha_1 - \beta_1),$$

in other words, in both cases,

$$\text{osc}_{\frac{1}{2}Q} u \leq (1 - \frac{1}{C})\text{osc}_Q u.$$

□

Proof of Corollary 6.13. For $s << 1$ we can write $\frac{1}{2^{\ell+1}} \leq s \leq \frac{1}{2^\ell}$. Using Theorem 6.12 we obtain,

$$\text{osc}_{\frac{1}{2^{\ell+1}}Q} u \leq \gamma \text{osc}_{\frac{1}{2^\ell}Q} u,$$

which implies, by induction that $\text{osc}_{\frac{1}{2^\ell}Q} u \leq \gamma^\ell \text{osc}_Q u$. Now, $2^{\ell+1} \geq \frac{1}{s}$ so that, $\ell \geq \frac{1}{\text{Log}\, 2}\text{Log}\, \frac{1}{s} - 1 = \rho(s)$ and, since $\gamma < 1$ we have, $\gamma^\ell \leq \gamma^{\rho(s)}$. Eventually since $sQ \subset \frac{1}{2^\ell}Q$ we obtain,

$$\text{osc}_{sQ} u \leq \gamma^{\rho(s)}\text{osc}_Q u.$$

We have just to notice that $\rho(s) \to +\infty$ when $s \to 0$ so $\tau(s) = \gamma^{\rho(s)} \to 0$ if $s \to 0$. □

6.2.5. *BMO norms of the eigenfunctions.* Here is a Corollary of Theorem 5.5.

Proposition 6.14. *There exists $C > 0$ such that for all φ_λ satisfying, $-\Delta_g \varphi_\lambda = \lambda \varphi_\lambda$ we have,*

$$\|Log\, |\varphi_\lambda|\|_{BMO} \leq C\sqrt{\lambda}.$$

Proof. Set $\psi_\lambda = \text{Log}\, |\varphi_\lambda|$, $(\psi_\lambda)_Q = \frac{1}{|Q|}\int_Q \psi_\lambda(x)\, dx$. Then,

$$\|\text{Log}\, |\varphi_\lambda|\|_{BMO} = \sup_Q I_Q \quad I_Q = \frac{1}{|Q|}\int_Q |\psi_\lambda(x) - (\psi_\lambda)_Q|\, dx. \tag{6.15}$$

Set $c_Q = \text{Log}\, \|\varphi_\lambda\|_{L^\infty(Q)}$ and $J_Q = \frac{1}{|Q|}\int_Q |\psi_\lambda(x) - c_Q|\, dx$. We have,

$$|(\psi_\lambda)_Q - c_Q| = \frac{1}{|Q|}\left|\int_Q (\psi_\lambda(x) - c_Q)\, dx\right| \leq \frac{1}{|Q|}\int_Q |\psi_\lambda(x) - c_Q|\, dx = J_Q.$$

It follows that,

$$I_Q \leq J_Q + \frac{1}{|Q|}\int_Q |(\psi_\lambda)_Q - c_Q|\, dx \leq 2J_Q.$$

We are lead to estimate J_Q.

We have seen in Theorem 4.3 that the doubling index of an eigenfunction corresponding to the eigenvalue λ is bounded by $C\sqrt{\lambda}$ for $\lambda \geq 1$.

Set for $(t, x) \in (0, 1) \times Q, u(t, x) = e^{t\sqrt{\lambda}}\varphi_\lambda$. Then u is a solution of the elliptic equation $(\partial_t^2 + \Delta_g)u = 0$. Moreover $\mathcal{N}_u(Q) \leq C\sqrt{\lambda}$. Theorem 5.5 implies that,

$$|\{(t, x) \in (0, 1) \times Q : |u(t, x)| \leq e^{-a} \sup_{(t,x)\in(0,1)\times Q} |u|\}| \leq Ce^{-\frac{\beta a}{\sqrt{\lambda}}}|Q|.$$

On the other hand, if $|\varphi_\lambda(x)| \leq e^{-a}\sup_Q |\varphi_\lambda|$ then, $|u(t, x)| \leq e^{-a}\sup_{(t,x)\in(0,1)\times Q} |u|$, so that,

$$\{x \in Q : |\varphi_\lambda(x)| \leq e^{-a}\sup_Q |\varphi_\lambda|\} \leq Ce^{-\frac{\beta a}{\sqrt{\lambda}}}|Q|.$$

Now, since $c_Q = \text{Log}\, \|\varphi_\lambda\|_{L^\infty(Q)}$ we have,

$$|\{x \in Q : |\varphi_\lambda(x)| \leq e^{-a}\sup_Q |\varphi_\lambda|\}| = |\{x \in Q : \theta(x) := c_Q - \psi_\lambda(x) \geq a\}| \leq Ce^{-\frac{\beta a}{\sqrt{\lambda}}}|Q|. \tag{6.16}$$

Therefore,

$$J_Q = \frac{1}{|Q|}\sum_{n=0}^{+\infty}\int_{\{x\in Q: n\sqrt{\lambda}\le\theta(x)\le(n+1)\sqrt{\lambda}\}}\theta(x)\,dx \le \frac{1}{|Q|}\sum_{n=0}^{+\infty}(n+1)\sqrt{\lambda}|\{x\in Q:\theta(x)\ge n\sqrt{\lambda}\}|.$$

Using (6.16) with $a = n\sqrt{\lambda}$ we get,

$$J_Q \le \sqrt{\lambda}\sum_{n=0}^{+\infty}(n+1)e^{-\beta n} \le C\sqrt{\lambda}.$$

□